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Mathematical Reviews

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ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Vicente Gonçalves, J. Sur quelques théorèmes classiques. Portugaliae Math. 3, 201–214 (1942). [MF 7908]

This article is devoted to the extension of the theorems of elementary calculus of Darboux, Rolle and Lagrange on continuous functions to discontinuous functions $f(x)$, defined in $I=(a, b)$, which are conditioned as follows: (1) f is a derived function; (2) it has a finite or infinite derivative at its points of continuity in the interior of I; (3) its points of discontinuity form a reducible set. *H. Blumberg.*

Kershner, Richard. The continuity of functions of many variables. Trans. Amer. Math. Soc. 53, 83–100 (1943). [MF 7826]

The main result extends, for the first time, to an arbitrary number n of variables a famous theorem of Baire hitherto proved only for the case $n=2$. This extension (which is here stated for the case $n=3$, as it can be done so more readily) is as follows. If $f(x_1, x_2, x_3)$ is a function, defined in the unit cube $(0\leqq x_i\leqq 1; i=1, 2, 3)$, which is continuous in each of the three variables, taken separately, there exists a residual set of planes parallel to each coordinate plane, on each of which there is a residual set of lines parallel to each possible coordinate axis consisting exclusively of points where f is continuous in its three arguments, taken together. A corollary, for the general case, is that the set of points where f is not continuous in all its n arguments, taken together, is of dimension not greater than $n-2$ (in the sense of Menger-Urysohn). A feature of the proof is the use of a new lemma, which generalizes a basic property observed by Baire. Let U_n be the unit cube in Euclidean n-space; a subset S of U_n is linearly closed if it intersects in a closed set every line parallel to a coordinate axis. Property Δ is defined inductively, with respect to n, for sets in U_n: a subset of U_1 has property Δ if it is of second category in U_1; a subset S of U_n has property Δ if there is a subset R of U_{n-1} such that R has property Δ in U_{n-1}, and, if $(x_1, x_2, \cdots, x_{n-1})$ belongs to R, then the set of numbers x_n for which $(x_1, x_2, \cdots, x_{n-1}, x_n)$ belongs to S is of second category. The aforementioned lemma is as follows. If $\{S_r\}$ is a sequence of sets in U_n such that S_r is linearly closed, and $\sum S_r$ has property Δ, then, for some k, S_k contains a subinterval of U_n. It is proved that the main theorem is the best such result obtainable. *H. Blumberg.*

Sard, Arthur. The measure of the critical values of differentiable maps. Bull. Amer. Math. Soc. 48, 883–890 (1942). [MF 7505]

The maps considered are the transformations (1) $y_j=f_j(x_1, x_2, \cdots, x_m)$, $j=1, 2, \cdots, n$, of a region R of Euclidean m-space into a subset of Euclidean n-space, the f_j being assumed to be of class C^q $(q\geqq 1)$ in R. A critical point of (1) is a point of R at which the matrix $\|\partial f_j/\partial x_i\|$ is of less than the maximum rank. A critical value is the image under (1) of a critical point. The principal results, which generalize known theorems, are the following, measure being taken in the sense of Hausdorff-Saks. If $m\leqq n$, the set of critical values of (1) is of m-dimensional measure 0 (without additional conditions on q); if $m>n$, this set is of n-dimensional measure 0 if $q\geqq m-n+1$. An example shows that this condition on q cannot be weakened. Change of variable, a variant of the Vitali theorem and a result of A. P. Morse are utilized in the reasoning. *H. Blumberg.*

Otchan, G. Quelques questions de l'équivalence des familles d'ensembles. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 6, 171–188 (1942). (Russian. French summary) [MF 7592]

Two families of subsets $\{A_\alpha\}$ and $\{B_\beta\}$ of sets E_1 and E_2, respectively, are called "equivalent" if there exists a one-to-one transformation of E_1 into E_2 such that each A_α is carried into some B_β and each B_β is the image of some A_α (the indices are from the same set R). The "bricks" of a family are the sets

$$K_z(A)=\prod_{\alpha\in z} A_\alpha \prod_{\beta\in z'} A_\beta',$$

where z is a subclass of R and primes denote complements. Two families $\{A_\alpha\}$ and $\{B_\beta\}$ are equivalent if and only if there exists a one-to-one transformation $\varphi(\alpha)$ of R into itself such that the cardinal numbers of $K_z(A)$ and $K_{\varphi(z)}(B)$ are equal. For denumerable families the author introduces a "function of mutual disposition" by associating with each x of Cantor's perfect set a brick K_x and defining $f(x)$ equal to the cardinal number of K_x. This function is investigated extensively and applied to obtain results (some due to E. Szpilrajn) concerning families of Borel sets and analytic sets. By a strong restriction of the families considered an analogue to the Cantor-Bernstein theorem is proved. Examples show the restrictions to be essential. A final section contains results previously announced [C. R. (Doklady) Acad. Sci. URSS (N.S.) 23, 753–755 (1939); these Rev. 1, 241]. *J. V. Wehausen* (Columbia, Mo.).

Moran, P. A. P. The measure of plane sets. Proc. Cambridge Philos. Soc. 39, 51–53 (1943). [MF 7969]

The author gives the latest in a long series of generalizations of a theorem first established by W. H. and G. C. Young [The Theory of Sets of Points, Cambridge, 1906, pp. 293 ff.]: the outer measure of a plane set of diameter d does not exceed $\pi d^2/4$. Let X be a bounded plane set, $P(\theta, X)$ its projection on the line $y=x\tan\theta$, and $f(\theta, X)$ the (outer) Lebesgue measure of $P(\theta, X)$. Let $M(X)=$l.u.b. $f(\theta, X)$ and let sm$X$ denote the outer two-dimensional Lebesgue measure of X. G. Szekeres [Acta Litt. Sci. Szeged 9, 247–252 (1940); these Rev. 1, 264] has shown that sm$X\leqq(\pi/4)[M(X)]^2$ when X consists of a finite number of continua. Moran shows that the same inequality holds when

X is any measurable set. The result can be extended to higher dimensions. Set $m=$g.l.b. $f(\theta, X)$ and

$$A=(1/2\pi)\int_0^{2\pi} f(\theta, X)d\theta.$$

When X consists of a finite number of continua, $f(\theta, X)$ is continuous and A exists, and Béla v. Sz. Nagy [Acta Litt. Sci. Szeged 9, 253–257 (1940); these Rev. 1, 264] has shown that $sm X \leqq (\pi/4)[A^2-(A-m)^2]$. Moran states that the same inequality holds for any measurable set X for which A exists, but errors invalidate his proof. His results show that A exists and that Nagy's inequality holds at least when X is closed. Moran proves that $f(\theta, X)$ is measurable when X is open, closed, or the sum of a sequence of closed sets. Examples show that his proof that $f(\theta, X)$ is measurable when X is the common part of a sequence of open sets is not valid. The first proof that $f(\theta, X)$ is measurable when X is closed was given by Mazurkiewicz and Saks [Fund. Math. 8, 109–113 (1926)]. *G. B. Price.*

Verčenko, I. Über das Flächenmass von Mengen. Rec. Math. [Mat. Sbornik] N.S. 10 (52), 11–32 (1942). (Russian. German summary) [MF 7766]

Let the surface $w(I_0)$, defined by the real continuous function $z=w(x, y)$ in the interval $I_0: a_1 \leqq x \leqq b_1; a_2 \leqq y \leqq b_2$, have finite Lebesgue area and let $|w(I)|$ denote the Lebesgue area of the curvilinear interval $w(I)$ corresponding to the subinterval I of I_0. Let now $\mathfrak{E}$ be an arbitrary point set on the surface $w(I_0)$, let $\mathfrak{E}^*$ be the orthogonal projection of $\mathfrak{E}$ on the x, y-plane and let $\{I_n\}$ be a finite or denumerable system of subintervals of I_0 covering $\mathfrak{E}^*$. Then the greatest lower bound of the set of numbers $\sum|w(I_n)|$ for all possible covering sets of intervals for the set $\mathfrak{E}^*$ is defined to be the outer superficial measure of the set $\mathfrak{E}$ relative to the surface $w(I_0)$ and is designated by $\Pi_w^*(\mathfrak{E})$. Using methods depending on the theory presented by S. Saks [Theory of the Integral, Stechert, New York, 1937], the author first shows that $\Pi_w^*(\mathfrak{E})$ satisfies the five Carathéodory characteristic properties of outer measure and then establishes the following principal result. If two surfaces of finite Lebesgue area are defined by $z=u(x, y)$, $z=v(x, y)$ in I_0, then for an arbitrary point set $\mathfrak{E}$ on the intersection of the surfaces we have $\Pi_u^*(\mathfrak{E})=\Pi_v^*(\mathfrak{E})$. *E. F. Beckenbach* (Austin, Tex.).

Kakutani, Shizuo. An extremum problem in product measure. Ann. of Math. (2) 43, 742–756 (1942). [MF 7405]

Let

$$A(\phi)=\int_0^1\int_0^1 \phi(x, y)dxdy,$$

$$V(\phi)=\int_0^1\int_0^1\int_0^1 \phi(x, y)\phi(y, z)dxdydz,$$

$0 \leqq \alpha \leqq 1$, $\lambda(\alpha)=\sup V(\phi)$ and $\mu(\alpha)=\inf V(\phi)$, where these extremes are taken over all real measurable functions ϕ defined on the square $0 \leqq x, y \leqq 1$ with $0 \leqq \phi(x, y) \leqq 1$ and $A(\phi)=\alpha$. It is shown that $\lambda(\alpha)=2\alpha-1+(1-\alpha)^{3/2}$ for $0 \leqq \alpha \leqq \frac{1}{2}$ and $\lambda(\alpha)=\alpha^{3/2}$ for $\frac{1}{2} \leqq \alpha \leqq 1$. Formulas for $\mu(\alpha)$ are also given. The extreme values of $V(\phi)$ are attained when ϕ is a characteristic function of certain sets which are determined explicitly. The problem of determining the functions $\lambda(\alpha)$ and $\mu(\alpha)$ was proposed by P. R. Halmos who, together with R. H. Fox, obtained the value of $\mu(\alpha)$ for $\alpha=\frac{1}{2}(1\pm 1/n)$, $n=1, 2, 3, \cdots$. *N. Dunford.*

Schwartz, H. M. Sequences of Stieltjes integrals. II. Duke Math. J. 10, 13–22 (1943). [MF 8097]

[The first part appeared in Bull. Amer. Math. Soc. 47, 947–955 (1941); these Rev. 3, 228.]

The paper is concerned with conditions under which the Riemann-Stieltjes integrals $\int_a^b f dg_n$ converge to $\int_a^b f dg$ (or zero) for all functions of a class F, which is a subclass of the class of functions for which all of the integrals involved exist. Solution of the problem for the case where F is the class of continuous functions and the g_n are of bounded variation is well known. The author considers the case where the functions of F are of bounded variation, which, by integration by parts, could have been reduced to the consideration of the integrals $\int g_n df$. The principal novel result is that, if g_n are of bounded variation and $\int f dg_n$ converges to zero for all functions f of F whose discontinuities of the second kind are restricted to a denumerable subset S of (a, b), it is necessary and sufficient that $g_n(x) \to 0$ for all common points of continuity of the g_n, that g_n be uniformly of bounded variation and the total variation functions $V_a^x g_n$ be equicontinuous at all points of S. *T. H. Hildebrandt.*

Jeffery, R. L. and Ellis, H. W. Cesàro totalization. Trans. Roy. Soc. Canada. Sect. III. (3) 36, 19–44 (1942). [MF 7880]

The paper develops a Denjoy integration based on a Cesàro mean. By a suitable definition of Cesàro derivatives, Burkill [Proc. London Math. Soc. (2) 34, 314–322 (1932); 39, 541–552 (1935)] succeeded in extending the Perron integral definition. The authors define a Cesàro totalization $CT(f, a, b)=CT\int_a^b f dx$ by starting, as Denjoy does, with $CT(f, E)=L\int_E f dx$ if f is Lebesgue integrable on E, but altering the definition for the case where $CT(f, \alpha, \beta)$ is defined for all $\alpha, \beta: \alpha_0<\alpha<\beta<\beta_0$ by replacing the limits

$$\lim_{h\to 0^+} CT(f, \alpha_0+h, \xi), \quad \lim_{h\to 0^+} CT(f, \xi, \beta_0-h)$$

by the Cesàro limits

$$\lim_{h\to 0^+} h^{-1}\int_{\alpha_0}^{\alpha_0+h} CT(f, \alpha, \xi)d\alpha=K(\alpha_0, \xi),$$

$$\lim_{h\to 0^+} h^{-1}\int_{\beta_0-h}^{\beta_0} CT(f, \xi, \beta)d\beta=K(\xi, \beta_0),$$

respectively, the functions under the integral sign being generalized Denjoy integrable and $K(\alpha_0, \xi)+K(\xi, \beta_0)$ being assumed independent of $\alpha_0<\xi<\beta_0$. By making assumptions paralleling those of Denjoy, the construction yields a definite function $F(x)$ if $F(a)=0$. This function $F(x)$ is generalized absolutely continuous on (a, b) and has an approximate derivative equal to $f(x)$ almost everywhere on (a, b). Additional conditions are needed so that $F(x)$ is C-continuous, that is,

$$\lim_{h\to 0^+} h^{-1}\int_x^{x+h} F(t)dt=F(x),$$

and so that the C derivative (CDF)

$$\lim_{h\to 0^+} 2h^{-2}\int_x^{x+h} [F(t)-F(x)]dt=f(x)$$

almost everywhere. If for a given $F(x)$ the CDF is finite everywhere on (a, b), then $CT(CDF, a, b)$ exists equal to $F(b)-F(a)$. A second order generalization is based on the mean $3h^{-3}\int_x^{x+h}(2x+h-2t)F(t)dt$. *T. H. Hildebrandt.*

Scott, William. A note on the lower semi-continuity of double integrals in the parametric form. Bull. Amer. Math. Soc. **48**, 763–767 (1942). [MF 7281]

The author refines a lemma of Radó [Trans. Amer. Math. Soc. **51**, 336–361 (1942); these Rev. **3**, 229] to obtain the slightly stronger lower semicontinuity theorem where the assumptions are $f \geqq 0$, $E \geqq 0$ (f is the integrand, E is the Weierstrass E-function). *M. Shiffman.*

Radó, Tibor. What is the area of a surface? Amer. Math. Monthly **50**, 139–141 (1943). [MF 8159]

Expository article.

Theory of Functions of Complex Variables

Garavito Armero, Julio. Elements of some entire functions. Revista Acad. Columbiana Ci. Exact. Fis. Nat. **5**, 65–68 (1942). (Spanish) [MF 8059]

The class of entire functions

$$(1) \qquad f_j(z) \equiv \sum_{n=0}^{\infty} z^{nj}/(nj)!,$$

where j is a positive integer, includes as special case the exponential $e^z \equiv f_1(z)$. Properties of the functions (1), suggested by properties of e^z, are investigated. One readily verifies that $f_j(z)$ coincides with its own jth derivative, and that this property is shared by the linearly independent functions $f_j'(z), f_j''(z), \cdots, f_j^{(j-1)}(z)$. Here then are j functions, a linear combination of which furnishes the general solution of the differential equation $d^n f(z)/dz^n = f(z)$. Let the jth roots of unity be $1, \omega, \omega^2, \cdots, \omega^{j-1}$. Then

$$(2) \qquad e^{\omega z} \equiv f_j(z) + \omega f_j^{(j-1)}(z) + \cdots + \omega^{j-1} f_j'(z), \quad j = 1, 2, 3, \cdots.$$

Now (2) is a generalization of the Euler identity (3) $e^{iz} \equiv \cos z + i \sin z$, since (2) reduces to (3) for $j=4$. We note that the author might equally well have based his developments on the following equivalent definition:

$$f_j(z) \equiv [e^z + e^{\omega z} + \cdots + e^{\omega^{j-1} z}]/j.$$

E. F. Beckenbach (Austin, Tex.).

Tschebotarëw, N. On some modification of Hurwitz's problem. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 223–226 (1942). [MF 7618]

A nonpolynomial entire function $f(z)$ is called a W-function if all its roots are on the imaginary axis. Let $f(z) = g(z) + ih(z)$, g and h real; $f(z)$ is called a WW-function if $f(z) + \epsilon h(z)$ is a W-function for every sufficiently small real ϵ. The author proves the following theorems: $f(z)$ is a W-function if and only if $\varphi(z) = g(z)/h(z) \neq -i$ in the lower half-plane. It is a WW-function if and only if the imaginary parts of $\varphi(z)$ and z are of opposite sign. Finally the question of when a W-function will also be a WW-function is studied. This is true for functions of order 0. However, it is not true for the function $\{\Gamma(iz)\}^{-1}$ of order 1. *G. Szegö.*

Mandelbrojt, S. Quasi-analyticity and properties of flatness of entire functions. Duke Math. J. **9**, 647–661 (1942). [MF 7920]

The principal theorem connects an even entire function $F(z)$ and an integrable function $f(x)$ of period 2π. Let $M(r)$ denote the maximum of $|F(re^{i\theta})|$; a_n, b_n are the Fourier coefficients of $f(x)$; it is assumed that $F(z) \not\equiv 0$ and that not all a_n, b_n vanish. Functions $I(\alpha)$, $\omega(\alpha)$ are defined by

$$I(\alpha) = -\log \int_{-\alpha}^{\alpha+\alpha} (|f(x)| + |f(-x)|)dx, \quad 0 < \alpha < 2\pi,$$

$$I(\alpha) = \omega[I(\alpha)/\alpha], \qquad I(\alpha)/\alpha \text{ large.}$$

The points of the (x, y) plane with coordinates

$$(\log n, -\log |F(n)(a_n + ib_n)|)$$

form a set E; $\Pi(x)$ denotes the largest convex function whose graph envelopes E from below. Then the equations

$$\lim_{r \to \infty} \frac{\log M(r)}{\omega(r)} = 0, \quad \lim_{\alpha \to +0} \frac{I(\alpha)}{\log \alpha} = -\infty$$

together imply that $\int^{\infty} \Pi(x) e^{-x} dx$ is convergent or $-\infty$. By taking special functions $F(z)$ corresponding to various hypotheses on $f(x)$, the author deduces sufficient conditions for various kinds of quasi-analyticity, previously obtained in other ways by Carleman, Ostrowski, de la Vallée-Poussin, himself [cf. Mandelbrojt, Séries de Fourier et classes quasi-analytiques de fonctions, Gauthier-Villars, Paris, 1935] and Lévine and Lifschetz [Rec. Math. [Mat. Sbornik] N.S. **9** (**51**), 693–711 (1941); these Rev. **3**, 106].

E. S. Pondiczery (Princeton, N. J.).

Vijayaraghavan, T. On functions represented by certain series. Quart. J. Math., Oxford Ser. **13**, 113–116 (1942). [MF 7635]

Extending a theorem of Pólya [Proc. London Math. Soc. (2) **21**, 36 (1922)], the author proves the following. Let $a_n \geqq 0$, $a_n = O(1)$ and $a_n \neq 0$ for infinitely many n. Then the unit circle $|z| = 1$ is natural boundary for the functions

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n z^n}{1 - z^n}, \quad g(z) = \sum_{n=1}^{\infty} \frac{a_n z^n}{1 + z^n}.$$

G. Szegö (Stanford University, Calif.).

Cotlar, Mischa. Functions which are univalent on a subset of the boundary of a domain of regularity. Publ. Inst. Mat. Univ. Nac. Litoral **4**, 47–96 (1942). (Spanish) [MF 7945]

Let $w = f(z)$, analytic in $|z| < 1$, remain continuous on an arc σ of length $l > 0$ on $|z| = 1$, and let D be the image of $|z| < 1$ on the w-plane. Then σ is said to be a proper arc if, for every point ξ on σ, the point $f(\xi)$ can be joined to the point at ∞ by a continuous curve containing no point of D; if, further, for all $\xi_1 \neq \xi_2$ on σ we have $f(\xi_1) \neq f(\xi_2)$, then $f(z)$ is said to be univalent on the proper arc σ. The above definitions and the following results may be extended by replacing σ by a set of positive measure on $|z| = 1$. For functions admitting proper boundary arcs of univalency, the author obtains several results which reduce to known theorems in the case $l = 2\pi$. Thus, if $n(l, r)$ is the number of zeros of $f(z)$ in $|z| < r < 1$, then

$$n(l, r) \leqq \pi \left[2 \tan^{-1} \left(\frac{1 - r}{1 + r} \tan \frac{l}{4} \right) \right]^{-1}.$$

Again, if $f(z)$ is analytic in $|z| < 1$ and continuous in $|z| \leqq 1$, then the Taylor series for $f(z)$ in $|z| < 1$ converges to $f(\xi)$ for almost all ξ on σ. *E. F. Beckenbach* (Austin, Tex.).

Gontcharoff, W. Sur une extension du théorème de Gauss-Lucas. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 39–41 (1942). [MF 7626]

Let C be a Jordan curve enclosing the domain D. A curve in D is called a "conformal straight line" if in the mapping

of D onto a circle it corresponds to a circular arc orthogonal to this circle. This leads in the customary way to the definition of "convex" domains F in D with respect to D. The least "convex" domain containing a given domain in D is then introduced in the usual manner. Now the following is true. Let $f(z)$ be analytic in D, $|f(z)|=\text{const.}$ on C. Let K be the set of the zeros of $f(z)$ in D. Then the zeros of $f'(z)$ are in the least "convex" domain containing K.

G. Szegö (Stanford University, Calif.).

Green, J. W. A special type of conformal map. Duke Math. J. **10**, 67–71 (1943). [MF 8102]

The following problem of conformal mapping is considered. Let G_z denote $|z|<1$, C_z denote $|z|=1$, E a proper subset of C_z open with respect to C_z and consisting of a finite or denumerable sum $\sum\alpha_i$ of open intervals α_i of C_z, F the complement of E with respect to C_z. It is proposed to find a conformal mapping $w(z)$ of G_z onto $|w|<1$ with certain radial slits removed, such that the points of F map continuously onto points of $|w|=1$ and the points of α_i map continuously onto the radial slits. This problem is equivalent to finding a Green's function for the mixed boundary value problem with respect to F and E.

It is assumed that $C_z-\bar{E}$ is not vacuous, and $C_z-\bar{E}$ is expressed as the sum $\sum\beta_i$ of disjoint open intervals β_i of C_z. The principal theorem of the paper is stated in terms of the class Γ of functions $f(z)$ which satisfy: (1) f is analytic and schlicht for $|z|<1$; (2) $|f|<1$ for $|z|<1$ and $f(0)=0$; (3) f is continuous on $G_z+(C_z-\bar{E})$ and $|f|=1$ on $C_z-\bar{E}$. This theorem is: there exists a function $g(z)$ in Γ such that $|g'(0)|\geqq|f'(0)|$ for every f in Γ. The mapping function $g(z)$ carries G_z into a region D_w of $|w|<1$, the boundary of D_w consisting only of points of $|w|=1$ and of certain radial slits. An interval β_i is mapped continuously on an arc β_i' of $|w|=1$. An interval α_i is mapped continuously onto all or part of one of the radial slits. If $\bar{E}-E$ is a set of capacity zero, then $g(z)$ is uniquely determined except for a factor of modulus one.

The family Γ is shown not to have zero as a limit function, by use of the Milloux-Nevanlinna inequality. The existence of the extremal function is then established by a combined use of Koebe's theorem and analytic continuation over the arcs β_i. Using a result of Löwner [Math. Z. **3**, 65–77 (1919)], the author shows that the extremal function has the stated properties on the intervals α_i and β_i. The assumption that $\bar{E}-E$ is of capacity zero implies the last statement of the theorem. *M. H. Heins* (Chicago, Ill.).

Bergman, S. et Marcinkiewicz, J. Sur les fonctions analytiques de deux variables complexes. J. Math. Phys. Mass. Inst. Tech. **21**, 125–141 (1942). [MF 7756]

This paper was to appear in Fund. Math. **33**, 75–94 (1939); it has been reviewed [these Rev. **1**, 10] from a reprint. It seems that volume **33** of the Fundamenta has not appeared.

Bergman, Stefan. A remark on the paper "Sur les fonctions analytiques de deux variables complexes." J. Math. Phys. Mass. Inst. Tech. **21**, 141–143 (1942). [MF 7757]

Let $\{\Psi_n(\varphi)\}$, $0\leqq\varphi\leqq2\pi$, be a set of continuous orthonormal functions which are complete in L^2. For each index n form the harmonic function $\psi_n(r,\varphi)$ possessing the boundary values $\Psi_n(\varphi)$ on the circle $r=1$. If $H(\varphi)$ is any function of L^2 and $a_n=\int_0^{2\pi}H(\varphi)\Psi_n(\varphi)d\varphi$, then the series $\sum a_n\psi_n(r,\varphi)$ converges absolutely for $r<1$ and approaches $H(\varphi)$ almost everywhere as r approaches unity. The author gives a very short proof of these results and then indicates how the procedure can be generalized to the case of two variables $z_1=r_1e^{i\varphi_1}$, $z_2=r_2e^{i\varphi_2}$ by use of the results of the paper cited in the preceding review. The result for two variables holds not only for the bicylinder $|z_1|<1$, $|z_2|<1$ but also for the general class of four dimensional domains possessing distinguished boundary surfaces. For such domains the notion of functions of the extended class would be used. (These functions are a generalization of double harmonic functions introduced by the author.) *W. T. Martin.*

Fourier Series and Generalizations, Integral Transforms

Šilov, G. E. On the Fourier coefficient of a class of continuous functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 3–7 (1942). [MF 7608]

The author considers a function $\psi(t)$ of period 2π, continuous, odd, such that $\psi(\pi-t)=\psi(t)$ and satisfying the following conditions: (i) $\psi(t)$ is increasing in $0\leqq t\leqq\pi/3$, and constant in $\pi/3\leqq t\leqq\pi/2$; (ii) $\psi'(t)$ is continuous and non-increasing in $0<t\leqq\pi/2$; (iii) $\lim t\psi'(t)/\psi(t)=0$ as $t\to0$. He proves that the Fourier series of $\psi(t)$ converges absolutely if and only if the series

$$\sum\frac{1}{2m+1}\psi\left(\frac{\pi}{2m+1}\right)$$

converges. From this result, applying a well-known theorem of Wiener, he deduces that, if a function $F(t)$ coincides in a neighborhood of the point $t=0$ with a function $\psi(t)$ of the above described type, and if the series $\sum n^{-1}F(\pi n^{-1})$ diverges, then the Fourier series of $F(t)$ does not converge absolutely. [The reviewer has not been able to follow the proof given for lemma 3 of the paper; he finds, however, that the lemma is correct.] *R. Salem* (Cambridge, Mass.).

Salem, R. On a theorem of Zygmund. Duke Math. J. **10**, 23–31 (1943). [MF 8098]

It is known that, if a function f of period 2π is of bounded variation, and its modulus of continuity $\omega(\delta)$ satisfies the condition

$$(*)\qquad \sum_1^\infty[\omega(1/n)]^{\frac{1}{2}}n^{-1}<+\infty,$$

then the Fourier series of f converges absolutely [see A. Zygmund, J. London Math. Soc. **3**, 194–196 (1928)]. The author now shows that the result fails if the exponent $\frac{1}{2}$ in (*) is replaced by any number greater than $\frac{1}{2}$.

A. Zygmund (South Hadley, Mass.).

Cheng, Min-Teh. The absolute convergence of Fourier series. Duke Math. J. **9**, 803–810 (1942). [MF 7930]

The following theorem may serve as a sample of the results obtained. If $1<p\leqq2$, $\epsilon>0$ and

$$\int_0^{2\pi}|f(x+h)-f(x)|^p dx=O(h|\log h|^{-p-\epsilon})\quad\text{as } h\to0,$$

then the Fourier series of f converges absolutely. The result fails for $\epsilon=0$. The arguments of the paper are mainly based on the inequalities of Hausdorff-Young. *A. Zygmund.*

Lepecki, Zbigniew. On some theorems on trigonometric double series. Fac. Filos. Ci. Let. do Paraná. Anuário. 1940–1941, 159–187 (1942). (Portuguese) [MF 7866]

The author extends Riemann theorems on localization to the case of double trigonometric series (S) $\sum_{m,n=-\infty}^{+\infty} c_{mn}e^{i(mx+ny)}$. The method is based on the formal multiplication of trigonometric series introduced by Rajchman [see, e.g., the reviewer's "Trigonometrical Series," Monografie Matematyczne, vol. 5, Warsaw, 1935, pp. 279 ff.] and extended by the author to the case of double series. The Riemann function $F(x, y)$ obtained by integrating the series (S) twice with respect to x and twice with respect to y also plays an important part. The coefficients c_{mn} are supposed either to tend to 0 in the classical Pringsheim sense (and be bounded) or to tend "strongly" to 0, which means that for every $\epsilon>0$ there may exist only a finite number of the c_{mn} exceeding ϵ in absolute value. To each of these assumptions corresponds a different set of theorems. The following theorem may be quoted as an example. Suppose that the Riemann functions $F(x, y)$ and $F_1(x, y)$ corresponding to trigonometric series S and S_1 with coefficients tending strongly to 0 are equal at the points of a "cross": $a\leqq x\leqq b$ or $c\leqq y\leqq d$. Then the series S and S_1 are uniformly equiconvergent in every rectangle $a+\epsilon\leqq x\leqq b-\epsilon$, $c+\epsilon\leqq y\leqq d-\epsilon$, $\epsilon>0$. Analogous results hold for the series $\sum c_{mn}\epsilon_m e^{i(mx+ny)}$, $\sum c_{mn}\epsilon_n e^{i(mx+ny)}$, $\sum c_{mn}\epsilon_m\epsilon_n e^{i(mx+ny)}$ ($\epsilon_m=-i$ sign m), which are, formally, conjugate to S with respect to x, to y and to both x and y. Ordinary equiconvergence must then be replaced by equiconvergence in a wider sense. Theorem 4′ contains an obvious slip of the translator: for "tends uniformly to zero" one should read "converges uniformly." [The paper, originally written in Polish, was scheduled to appear in the June, 1939 issue of the Polish quarterly "Wiadomości Matematyczne," and was already available in the reprint form. An early appearance of that issue seems unlikely now.] *A. Zygmund* (South Hadley, Mass.).

Guinand, A. P. Simple Fourier transformations. Quart. J. Math., Oxford Ser. 13, 153–158 (1942). [MF 7914]

If α_r, A_r are constants, $\Re(\alpha_r)\neq\frac{1}{2}$ and β_r are nonnegative integers, $r=1, \cdots, n$, such that

$$\prod_{r=1}^{n}\frac{1-s-\alpha_r}{s-\alpha_r}\equiv 1+\sum_{r=1}^{n}\frac{A_r}{(s-\alpha_r)^{\beta_r+1}},$$

and if $\gamma_r=1$ for $\Re(\alpha_r)>\frac{1}{2}$ and $\gamma_r=0$ for $\Re(\alpha_r)<\frac{1}{2}$, then the square of the transformation

$$Tf=x^{-1}f(x^{-1})+\sum_{r=1}^{n}(-1)^{\gamma_r}(A_r x^{-\alpha_r}/\beta_r!)\times\int_0^{x-1}t^{-\alpha_r}(-\log xt)^{\beta_r}f(t)dt$$

is the identity, for $x^{-\frac{1}{2}}f(x)$ belonging to $L(0, \infty)$. The statement can be generalized to functions $f(x)$ in several variables. *S. Bochner* (Princeton, N. J.).

Kendall, D. G. A summation formula associated with finite trigonometric integrals. Quart. J. Math., Oxford Ser. 13, 172–184 (1942). [MF 7916]

Let $f(x)$ be a function defined over the interval $(0, \infty)$. The main result of the paper is that, under suitable restrictions on f, a necessary and sufficient condition that the summation formula

$$(*)\qquad \tfrac{1}{2}\lambda f(0)+\lambda\sum_{n=1}^{\infty}f(n\lambda)=\int_0^{\infty}f(x)dx$$

should be valid for all λ in some interval $0<\lambda<\lambda_1$ is that the cosine transform

$$F_c(t)=(2/\pi)^{-\frac{1}{2}}\int_0^{\infty}f(x)\cos xt\,dx$$

should vanish for $t>2\pi/\lambda_1$. More precisely, if $f(x)$ tends to 0 for $x\to+\infty$, and both f and f' are of bounded variation over $(0, \infty)$, then both sides of (*) are equiconvergent for all $0<\lambda<\lambda_1$ if and only if $F_c(t)=0$ for all $t>2\pi/\lambda_1$. It is also shown [the result is due to Titchmarsh] that, if the integral function $\varphi(z)$ is $O(e^{a|z|})$ as $|z|\to+\infty$ and if on the real axis $\varphi(x)\to 0$ as $|x|\to+\infty$, then

$$\lim_{N\to\infty}\left\{\lambda\sum_{n=-N}^{+N}\varphi(n\lambda)-\int_{-(N+\frac{1}{2})\lambda}^{(N+\frac{1}{2})\lambda}\varphi(x)dx\right\}=0$$

for all $0<\lambda<2\pi/a$. *A. Zygmund* (South Hadley, Mass.).

Barnard, R. W. and Goldstine, H. H. The modular space determined by a positive function. Bull. Amer. Math. Soc. 48, 946–948 (1942). [MF 7518]

If V is a monotone increasing bounded function in $(-\infty, \infty)$, the modular functions of E. H. Moore in $(-\infty, \infty)$ relative to the base matrix $\epsilon(x, y)=\int_{-\infty}^{\infty}e^{i(x-y)t}dV(t)$ are precisely the functions $\mu(x)=\int_{-\infty}^{\infty}e^{ixt}\lambda(t)dV(t)$, with λ measurable and λ^2 integrable with respect to V. Also, $N\mu=\int_{-\infty}^{\infty}|\lambda|^2dV$ and $J\mu_1\mu_2=\int_{-\infty}^{\infty}\bar{\lambda}_1\lambda_2 dV$. *S. Bochner.*

Gross, Bernhard. On an integral transformation of general circuit theory. Amer. Math. Monthly 50, 90–93 (1943). [MF 8051]

This is a résumé of the results obtained by the author, B. Levi and L. Gama [Anais Acad. Brasil. Ci. 12, 317–318 (1940); 13, 31–50, 51–55, 163–164, 185–194 (1941); Publ. Inst. Mat. Univ. Nac. Litoral 3, 117–120, 121–129 (1941); these Rev. 4, 98] concerning the transformation

$$\pi f(s)=2\int_0^{\infty}s(s^2-t^2)^{-1}F(t)dt.$$

Among the topics discussed are the existence of the transformation, the orthogonality of the kernel, the inversion of the transformation and the transform of a Fourier series. *I. Opatowski* (Chicago, Ill.).

Differential Equations

Vicente Gonçalves, J. Sur une formule de récurrence. Portugaliae Math. 3, 215–233 (1942). [MF 7981]

The author considers polynomial solutions of the equations $e^n(y)=0$ ($n=0, 1, 2, \cdots$), where

$$e^n(y)\equiv(a_0x^2+a_1x+a_2)y''+(b_0x+b_1)y'-n[a_0(n-1)+b_0]y,$$

and examines in detail the conditions under which these equations have polynomial solutions of degree n and lower. Among the results established are that, for n sufficiently large, say $n>n_0$, the equation $e^n(y)=0$ has no polynomial solutions of degree lower than n, but has a polynomial solution of degree equal to n. The principal result of the paper is that there exists a recurrence relation of the form

$$y_{n+1}=(x+\alpha_n)y_n+\beta_n y_{n-1},\qquad n>n_0,$$

where y_i is the polynomial solution of degree i and leading coefficient equal to 1 of the equation $e^i(y)=0$, and α_n and

β_n are constants. Formulas for determining n_0, α_n and β_n are developed. The paper has a number of typographical errors, especially among the illustrative examples.

D. Moskovitz (Pittsburgh, Pa.).

Boas, Mary L., Boas, R. P., Jr. and Levinson, Norman. The growth of solutions of a differential equation. Duke Math. J. 9, 847–853 (1942). [MF 7934]

Relative to the differential equation $y''(x)+A(x)y(x)=B(x)$, in which the coefficients are continuous in $0\leqq x<\infty$, the authors establish the fact that $\lim_{x\to\infty} y'(x)$ exists for every solution $y(x)$, under either of the sets of conditions which follow:

(1) $\int_0^\infty xA(x)dx<\infty$, and $\int_0^\infty B(x)dx$ exists,

(2) $\limsup_{x\to\infty}\int_a^x xA_1(x)dx<1$, with $A_1(x)=\frac{1}{2}[|A(x)|+A(x)]$,

$\int_0^\infty xA_2(x)dx<\infty$, with $A_2(x)=\frac{1}{2}[|A(x)|-A(x)]$,

and

$$\int_0^\infty B(x)dx \text{ exists.}$$

Under the conditions (2) it is shown furthermore that $\lim_{x\to\infty} y'(x)=0$ if

$$\int_0^\infty xA_1(x)dx=\infty.$$

R. E. Langer (Madison, Wis.).

Pugachov, V. S. Evaluation of error of asymptotic representations of integrals of linear differential equations containing a parameter. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 203–208 (1942). (Russian. English summary) [MF 7750]

The paper is concerned with the equation

$$(1)\qquad \frac{d^2y}{dx^2}+2\alpha p(x,\alpha)\frac{dy}{dx}+\alpha^2 q(x,\alpha)=\alpha^r e^{\alpha\varphi(x)}F(x,\alpha),$$

$$a\leqq x\leqq b,\quad \alpha>0,$$

where

$$p(x,\alpha)=p_0(x)+\frac{p_1(x)+p_2(x,\alpha)}{\alpha},$$

$$q(x,\alpha)=q_0(x)+\frac{q_1(x)+q_2(x,\alpha)}{\alpha},$$

$$F(x,\alpha)=F_0(x)+\frac{F_1(x)+F_2(x,\alpha)}{\alpha},$$

with $q_0-p_0^2>0$, $R[\varphi(x)+\int_a^x p_0(x)dx]\equiv 0$, and p_2, q_2, F_2 approaching zero uniformly, relative to x, as $\alpha\to\infty$. (In addition the functions p_i, q_i, F_i, φ are supposed to have derivatives of sufficiently high orders.) From the well-known results of G. D. Birkhoff and Fowler-Lock, generalized by the author [Pougatcheff, W., Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 75–84 (1941); these Rev. 2, 288], one obtains the following asymptotic representations for a particular integral y of (1) and for two linearly independent integrals y_1, y_2 of the corresponding homogeneous equation:

$$(2)\qquad y\approx\alpha^{r-2}e^{\alpha\varphi(x)}\frac{F_0}{q_0+\varphi'^2+2p_0\varphi'},$$

$$(3)\qquad y_1,\ y_2\approx\frac{1}{\sqrt[4]{q_0-p_0^2}}\exp\left\{-\alpha\int_a^x p_0dx \pm i\int_a^x\left[\alpha\sqrt{q_0-p_0^2}+\frac{q_1-2p_0p_1-p_0'}{2\sqrt{q_0-p_0^2}}\right]dx\right\}.$$

The purpose of the present paper is to evaluate the errors in the asymptotic formulas (2) and (3). The results are too complicated to be stated here. The author remarks that his evaluations have been used by him in investigating trajectories of spinning shells and vibrations of aircraft bombs.

W. Hurewicz (Providence, R. I.).

Nagabhushanam, K. Infinitesimal automorphisms of the action form. Math. Student 10, 64–67 (1942). [MF 7953]

Let the infinitesimal automorphisms of $A_d=p_r dq^r-Hdt$, where H is the Hamiltonian, be denoted by $\delta q^r=\varphi\xi^r\delta t$, $\delta p_r=\varphi\eta_r\delta\tau$, where φ is arbitrary, $r=1, 2, \cdots, n$. Then the equations

$$\eta_k=-\frac{\partial f}{\partial q^k}+H\frac{\partial\zeta}{\partial q^k},\quad \xi^k=\frac{\partial f}{\partial p_k}-H\frac{\partial\zeta}{\partial p_k}$$

exist, where $f-H\zeta=0$. These equations form the most general solution to the question of the infinitesimal automorphisms of A. A necessary and sufficient condition that A_d admits these automorphisms is that H be explicitly independent of t. The paths of the automorphisms of A_d cannot include the dynamical trajectories. The method followed is that of L. P. Eisenhart [Ann. of Math. (2) 30, 211–249 (1929)].

D. J. Struik (Cambridge, Mass.).

Gunther, N. Sur les modules des formes algébriques. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 9, 97–206 (1941). (French. Russian summary) [MF 7381]

The expressions $F_i=a_1\varphi_i^{(1)}+a_2\varphi_i^{(2)}+\cdots+a_s\varphi_i^{(s)}$, where the a_j are arbitrary numbers and the $\varphi_i^{(j)}$, $j=1, 2, \cdots, s$; $i=1, 2, \cdots, z$, are homogeneous polynomials of dimension $n^{(j)}$ in $x_1, x_2, \cdots, x_m$, are called generalized forms in $x_1, x_2, \cdots, x_m$. Each such form is a linear homogeneous function of the products $a_\rho x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m}$, $\alpha_1+\alpha_2+\cdots+\alpha_m=n^{(j)}$. The F_i are called linearly independent if they are linearly independent with respect to $a_\rho x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m}$. Now the generalized forms $B_1^{(k)}F_1+B_2^{(k)}F_2+\cdots+B_z^{(k)}F_z$, $k=1, 2, \cdots$, are formed. The linearly independent forms among them are called the modul of the algebraic forms of base F_i. Several problems in connection with these forms are solved, among them the determination of the linearly independent forms $\sum B_i^{(k)}F_i$ for every given k, and the formation, for every given k, of all identical relations of the form $\sum B_i^{(k)}F_i=0$. Every such identity is a condition of passivity of the forms F_i. For $s=1$ these problems have been solved by Hilbert. In this paper the general problem is taken up in connection with the theory of systems of partial differential equations as developed by Janet [Mémor. Sci. Math., v. 21, Gauthier-Villars, Paris, 1927], which also gives a new approach to Hilbert's case. Applications are made to Riquier's theory, to the treatment of elimination problems and to the improvement of the theory of Delassus. The paper follows the trend of thought of the author's paper in the Proceedings of the first Allrussian Mathematical Congress [Kharkov, 1930, pp. 240–253].

D. J. Struik.

Nielsen, K. L. and Ramsay, B. P. On particular solutions of linear partial differential equations. Bull. Amer. Math. Soc. **49**, 156–162 (1943). [MF 8000]

For the differential equation

(1) $$U_{z\bar{z}}+aU_z+bU_{\bar{z}}+cU=0, \quad z=x+iy;\ \bar{z}=x-iy,$$

S. Bergman [Rec. Math. [Mat. Sbornik] N.S. **2** (**44**), 1167–1197 (1937)] has shown there exist functions $E(z, \bar{z}, t)$ such that

$$P(f)=\int_{-1}^{+1} E(z, \bar{z}, t)f(z(1-t^2)/2)dt/(1-t^2)$$

will be a solution of (1) for each analytic function $f(z)$. This paper presents and discusses five choices of the coefficients of (1) such that E will take the form

$$E(z, \bar{z}, t)=\exp(Nt^n+Mt^m), \quad N=N(z, \bar{z});\ M=M(z, \bar{z}).$$

F. G. Dressel (Durham, N. C.).

Bergman, Stefan. Linear operators in the theory of partial differential equations. Trans. Amer. Math. Soc. **53**, 130–155 (1943). [MF 7828]

Denote by $C(1)$ the totality of analytic functions $f(z)$ regular at the origin. The author investigates transformations $u(z, \bar{z})=P(f)$, $f\varepsilon C(1)$, where $P(f)$ is an operator defined by

$$P(f)=\int_{-1}^{1} E(z, \bar{z}, t)f[(z/2)(1-t^2)](1-t^2)^{-\frac{1}{2}}dt$$

and $E(z, \bar{z}, t)$ satisfies certain very general conditions. The class of functions $u(z, \bar{z})=P(f)$ for $f\varepsilon C(1)$ is denoted by $C(E)$. The conditions on E are such that $u(z, \bar{z})=P(f)$ is an analytic function of the two complex variables $z, \bar{z}$. The author first obtains a simple form of the operator $R(\zeta\,|\,u)$ inverse to $P(f)$ and shows that, if $u(z, \bar{z})$, $u\varepsilon C(E)$, is regular in a star-domain F of the z-plane, then $f(z/2)=R(z/2\,|\,u)$ is also regular in F. These results depend upon the values of $u(z, \bar{z})$ in a four-dimensional domain containing F. He also shows how f may be obtained in terms of $u(z, \bar{z})$ in the real plane, that is, for $z, \bar{z}$ conjugates. Various other results are obtained including some results which show a strong duality between the theory of analytic functions of a complex variable and the theory of functions of $C(E)$. For example, an integral formula, series developments and approximation results are obtained.

The study of the class $C(E)$ is of interest in the theory of linear partial differential equations of the elliptic type. In fact, the author has previously shown that for every equation

$$L(U)=U_{z\bar{z}}+a(z, \bar{z})U_z+b(z, \bar{z})U_{\bar{z}}+c(z, \bar{z})U=0,$$

$U_z=\partial U/\partial z$, etc., there exist functions $E_j(z, \bar{z}, t)$, $j=1, 2$, such that every solution of $L(U)=0$ can be written in the form

$$\left[\exp\left(-\int_0^{\bar{z}} a\,d\bar{z}\right)\right]\cdot C(E_1)+\left[\exp\left(-\int_0^{z} b\,dz\right)\right]\cdot C^*(E_2),$$

where $C^*(E)$ is a class analogous to $C(E)$, the associates of whose functions are analytic functions of $\bar{z}$.

W. T. Martin (Cambridge, Mass.).

Bergman, Stefan. Residue theorems of harmonic functions of three variables. Bull. Amer. Math. Soc. **49**, 163–174 (1943). [MF 8001]

Let F be the class of analytic functions of complex variables u and ζ, where $u=x_1+(i/2)(x_2+ix_3)\zeta^{-1}+(i/2)(x_2-ix_3)\zeta$. Denote by C a closed curve in the ζ-plane, and by $N(\mathbf{T})$ a sufficiently small neighborhood of a point $\mathbf{T}=(t_1, t_2, t_3)$. Then the operation $P(f, C, \mathbf{T})=(2\pi i)^{-1}\int_C f(u, \zeta)d\zeta$, for $\mathbf{X}=(x_1, x_2, x_3)\varepsilon N(\mathbf{T})$, carries F into the class of (complex) harmonic functions of three real variables x_1, x_2, x_3. For a given f the harmonic function so obtained depends of course upon the point $\mathbf{T}$ in the neighborhood of which the integration is carried out. Various properties of the function f of the integrand are reflected in the harmonic function $P(f, C, \mathbf{T})$. For example, the operation P transforms a polynomial in u and ζ into a harmonic polynomial of the same degree, and in general it transforms a rational function f into an algebraic harmonic function. The author considers in some detail the rational case $f=p/q$, where p and q are polynomials in u and ζ. A general residue formula for line integrals is obtained by integrating the various branches of an algebraic harmonic vector $\mathbf{S}(\mathbf{X})$ over appropriate curves in (x_1, x_2, x_3)-space. The representation P enables one to use the residue theorem for functions of one complex variable. The paper closes with some residue formulas for surface integrals, and a generalization of Cauchy's formula.

W. T. Martin (Cambridge, Mass.).

Brachkovsky, B. Z. Evaluation of the accuracy of the variation methods of solution of Poisson's equation. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 489–490 (1941). (Russian. English summary) [MF 7729]

Let $u(x, y)$ be the unknown solution of the Poisson equation $\Delta u=-p(x, y)$, with boundary condition $u=\varphi(x, y)$, for the plane region R, and let $u^*(x, y)$ be an approximate solution satisfying the boundary condition exactly but with $\Delta u^*=-p^*(u, v)$; let $f(x, y)=p(x, y)-p^*(x, y)$. Then Green's theorem gives

$$u-u^*=-(2\pi)^{-1}\int \delta_R f G d\Omega,$$

where $G(x, y; \xi, \eta)$ is the Green's function for R with pole at (ξ, η). A simple bound on the difference $u-u^*$ follows immediately in terms of $\max|f|$ and the diameter of R.

E. F. Beckenbach (Austin, Tex.).

Behrbohm, H. und Pinl, M. Zur Theorie der kompressiblen Potentialströmungen. II. Intermediäre und singuläre Integrale der Grundgleichung der ebenen adiabatisch kompressiblen Potentialströmung. Z. Angew. Math. Mech. **21**, 341–350 (1941). [MF 7658]

[Part I appeared in the same **Z. 21**, 193–203 (1941).]

The velocity potential for the two dimensional adiabatic stationary potential flow of a compressible fluid satisfies the well-known equation

(1) $$(\lambda+\mu(\phi_x^2+\phi_y^2)-\phi_x^2)\phi_{xx}-2\phi_x\phi_y\phi_{xy}+(\lambda+\mu(\phi_x^2+\phi_y^2)-\phi_y^2)\phi_{yy}=0,$$

where λ and μ are certain constants. (For orientation, it may be mentioned that, for $\lambda=\mu=1$, (1) is the minimal surface equation.) The usual treatment of (1) is that of transformation to a linear equation by a Legendre transformation. The authors, on the other hand, try the direct approach of the method of characteristics. The bicharacteristic equations of (1) involve ϕ_x and ϕ_y in two integrable combinations (2) $F_i(\phi_x, \phi_y)=0$, $i=1, 2$. These are first order equations which themselves can be treated by the characteristics method. The authors treat the singular case $F_1=F_2$ by the theory of the complete integral. The investigation amounts to an exercise in the straightforward application of standard methods to the particular equation (1).

D. G. Bourgin (Urbana, Ill.).

Friedlander, F. G. On the solutions of the wave equation with discontinuous derivatives. Proc. Cambridge Philos. Soc. 38, 378–382 (1942). [MF 7805]

Consider the wave equation and the wave front $T=t-f(x, y, x)=0$. If $\sigma=\partial u/\partial T|_{0-}^{0+}$ is well behaved, then it is known that (1) $2\partial\sigma/\partial\nu+\sigma\nabla^2 f=0$, where ν is the wave normal. Hence (2) $\nabla\cdot\sigma^2\nabla f=0$. The simple step of going from (1) to (2) leads to a pretty conclusion, for (2) implies (3) $\iint_S\sigma^2\cos\theta dS=0$, where S is closed and θ is the angle between the surface and wave normals at points on S. Consider S_1, a piece of S bounded by a simple closed curve, and S_2, the corresponding surface piece after a time interval t. Then (3) gives $\iint_{S_1}\sigma^2 dS=\iint_{S_2}\sigma^2 dS$. This is the conclusion sought, which, surprisingly enough, has been overlooked heretofore. The author indicates also the behavior of the saltus in transversing points on the caustic.

D. G. Bourgin (Urbana, Ill.).

Friedlander, F. G. On the reflexion of a spherical sound pulse by a parabolic mirror. Proc. Cambridge Philos. Soc. 38, 383–393 (1942). [MF 7806]

If ϕ is the velocity potential associated with "small amplitude" irrotational sound propagation, with the sound velocity taken as 1, then in proper units (1) $\partial\phi/\partial t=p$, where p is the pressure increment. Interpret the scripts i and R as incident and reflection values. Consider the pulse (2) $\phi_i=r^{-1}S_1(t-r)$ emanating from the origin, where $S_1=0$ for nonpositive arguments. Let $r=x+2$ define a reflecting paraboloid of revolution Γ with focus at the origin. The boundary conditions are (3) $(\partial/\partial n)(\phi_i+\phi_R)|_\Gamma=0$ and (4) $\phi_R|_{t=r}=0$. Write $T=t-(x+1)$ [a trivial slip introduces a constant whose value is 1] and define $S_k(T)$ inductively by $\int_0^T S_{k-1}(z)dz$. A formal solution may be written as (5) $\phi_R=\sum_{k=1}^\infty\psi_k(x,\rho)S_k(T)$ with $\rho^2=x^2-r^2$. On substituting in the wave equation there results the recurrence scheme $2\partial\psi_k/\partial x=\nabla^2\psi_{k-1}$, $\psi_0=0$, subject to (3). Apparently the representation (5) is used because (4) is then automatically satisfied. The greater part of the paper concerns itself with explicit computation of the first few ψ's and some numerical evaluations, with a view to investigating p. Unfortunately no quantitative conclusions can be inferred, for the series (5) converges "slowly" (if at all) away from the vertex and a study of the convergence situation here seems essential, as the author himself admits. *D. G. Bourgin.*

Mersman, W. A. Heat conduction in an infinite composite solid with an interface resistance. Trans. Amer. Math. Soc. 53, 14–24 (1943). [MF 7822]

Let $U(x, t)$ denote the temperature function for a solid composed of a material with thermal diffusivity a_1 in the region $x<0$ and another material with diffusivity a_2 in the region $x>0$. At the interface $x=0$ there is thermal resistance with a coefficient λ. The function $U(x, t)$ then satisfies the partial differential equation $U_t=a_\nu U_{xx}$, where $\nu=1$ when $x<0$ and $\nu=2$ when $x>0$, and the boundary conditions $U(x, +0)=f(x)$ $(x\neq 0)$ and $k_1U_x(-0, t)=k_2U_x(+0, t)$, $U(+0, t)-U(-0, t)=\lambda k_1U_x(-0, t)$, where k_1 and k_2 are the thermal conductivities of the two materials. The solution of this problem is derived and then verified in full, by means of the Laplace transformation, when $f(x)$ is any integrable function of exponential order. The formula for $U(x, t)$ involves infinite integrals of products of f and certain exponential functions. *R. V. Churchill.*

Integral Equations

Muskhelišvili, N. I. Application of integrals of Cauchy type to a class of singular integral equations. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 10, 1–43 (1941). (Russian. Georgian summary) [MF 7383]

The author gives a method, based on properties of integrals of Cauchy type, for investigating singular integral equations of forms

(A) $$\frac{1}{\pi i}\int\frac{k(t_0, t)\varphi(t)dt}{t-t_0}=f(t_0)$$

and

(B) $$\alpha(t_0)\varphi(t_0)+\frac{1}{\pi}\int\frac{k(t_0, t)\varphi(t)dt}{t-t_0}=f(t_0),$$

where integration in the sense of principal Cauchy values is over a set L of open curves; $k(t_0, t)$, $\alpha(t)$, $f(t)$ are assigned functions of Hölder type. The treatment of (B) is based to some extent on Carleman's methods [Ark. Mat. Astr. Fys. 16, no. 26 (1922)] and has connection with a paper by Vecoua [C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 327–330 (1940); these Rev. 2, 312]. The following definitions are given: $\varphi(t)\subset H$ (with index λ) if (1) $|\varphi(t'')-\varphi(t')|\leqq A|t''-t'|^\lambda$ $(\lambda\leqq 1)$ in the interior of each one of the arcs of L; $\varphi(t)\subset H_0$ if (1) holds including end points of the arcs; $\varphi(t)\subset H^*$ if $\varphi(t)\subset H$, while in the vicinity of each end point c one has $\varphi(t)=(t-c)^{-\alpha}\varphi^*(t)$ $(0<\alpha<1)$, where φ^* is of Hölder type in the vicinity and at c. A typical theorem is as follows. Given $f(t)\subset H^*$ and such that near end points c one has $f(t)=(t-c)^{-\alpha}f^*(t)$ $(0\leqq\alpha<1)$, while Hölder's index of $f^*(t)$ exceeds $\alpha-\frac{1}{2}$, the equation

$$\int_L\varphi(t)dt/(t-t_0)=\pi i f(t_0), \qquad t_0 \text{ on } L,$$

has a solution $\varphi(t)\subset H^*$. Interesting applications are given to certain boundary problems of Dirichlet and Riemann types. *W. J. Trjitsinsky* (Urbana, Ill.).

Muskhelišvili, N. I. Remark to "Application of integrals of Cauchy type to a class of singular integral equations." Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 10, 161–162 (1941). (Russian) [MF 7388]

Correction of an oversight in the paper reviewed above.

W. J. Trjitsinsky (Urbana, Ill.).

Vecoua, I. Integral equations with a singular kernel of Cauchy type. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 10, 45–72 (1941). (Russian. Georgian summary) [MF 7384]

This work unifies some of the earlier results of the author. On the basis of a new method there is presented a systematic theory of equations

(A) $$A\varphi\equiv\alpha(x)\varphi(x)-\int_L\frac{k(x, t)}{t-x}\varphi(t)dt=f(x);$$

L is the boundary of a multiply connected region T, bounded by a finite number of simple contours $L_0, \cdots, L_m$ without common points, L_0 containing $L_1, \cdots, L_m$; $\alpha(x)$, $k(x, t)$, $f(x)$ are given complex-valued functions of points on L; $\varphi(x)$ is the unknown. Of importance is the characteristic equation

(B) $$B\varphi\equiv\alpha(x)\varphi(x)-\beta(x)\int_L\varphi(t)dt/(t-x)=f(x),$$

where $\beta(x)=k(x, x)$. It is assumed that $\alpha(x)$, $k(x, t)$, $f(x)\subset H$ (that is, are of Hölder class); $\alpha^2(x)+\pi^2\beta^2(x)\neq 0$ on L; the

coordinates of points on the L_j have first order arc-derivatives of Hölder class. On letting

$$2\pi n_j = \left\{ \arg \frac{\alpha(x)+i\pi\beta(x)}{\alpha(x)-i\pi\beta(x)} \right\}_{L_j},$$

the index n is defined as $n_0+\cdots+n_m$. Some of the results are as follows. If $n=0$, $B\varphi=f$ has a unique solution of Hölder class. If $n>0$, $B\varphi=f$ has ∞^n solutions of Hölder class. If $n<0$ then "generally" $B\varphi=f$ has no solutions. If $n=0$, $A\varphi=f$ is equivalent to a Fredholm equation. If $n>0$, $A\varphi=f$ is equivalent to a Fredholm equation involving n arbitrary constants. When $n<0$, the problem of solvability of $A\varphi=f$ is equivalent to that for a system of functional equations of Fredholm type. The number of linearly independent H-solutions of $A\varphi=0$ is finite. The author also gives very simple proofs of two theorems of F. Noether [Math. Ann. **82**, 42–63 (1921)]. *W. J. Trjitzinsky.*

Vecoua, N. P. and Kwesselawa, D. A. On a boundary value problem of the theory of functions of a complex variable and its application to the solution of a system of singular integral equations. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **9**, 33–48 (1941). (Russian) [MF 7377]

The authors give an elementary solution of Riemann's linear boundary problem, with an application to regularization of a singular system of integral equations (in which integrals are in the sense of principal Cauchy values). Here L denotes a simple closed contour; $S^+(S^-)$ is the domain interior (exterior) to L; $\varphi(x)$, defined in S^+, is c. (continuable) at x on L from the side S^+ if $\lim \varphi(z)$ exists when $z\to x$ from within S^+; there is a similar definition for $\varphi(x)$ c. at x on L from the side S^-. Following Mushelišvili, a function analytic everywhere except on L and c. at every point of L from both sides is termed p.a. (piecewise analytic); p.a. functions are denoted by $\varphi(z)$; $\varphi^+(z)(\varphi^-(z))$ will be the function in $S^+(S^-)$. On L one writes $\varphi^+(x)=\lim \varphi^+(z)$, as $z\to x$, etc. The Riemann problem solved by the authors is the following. Given $a_{ik}(x)$, $b_i(x) \subset H$ (that is, of Hölder class) on L, $\det |a_{ik}| \neq 0$, find n p.a. functions $\varphi_i(z)$ such that, on L,

$$\varphi_j^+(x)=\sum_k a_{jk}(x)\varphi_k^-(x)+b_j(x).$$

One of the singular integral systems considered by the authors is

$$\text{(A)} \qquad \alpha_j(x)\varphi_j(x)+\sum_{k=1}^{n}\frac{\beta_{jk}(x)}{\pi i}\int_L \frac{\varphi_k(t)dt}{t-x}=f_j(x),$$

coefficients of Hölder class on L, $j=1, \cdots, n$; they associate with (A) a certain Riemann problem (B) $\Phi_i^+=\sum_k A_{ik}\Phi_k^-+\beta_i$ and obtain solution of (A) in the form $\varphi_i(x)=\Phi_i^+(x)-\Phi_i^-(x)$. It is shown how the system (A) may be replaced by a single integral equation (with Cauchy principal values); the regularization of the latter has been carried out completely [Vecoua, Mushelišvili; cf. the papers reviewed above].

W. J. Trjitzinsky (Urbana, Ill.).

Kuznetsov, E. S. Scattering of light in a medium bordering on a reflecting wall with given albedo. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1942**, 228–242 (1942). (Russian. English summary) [MF 8088]

The author calculates the intensity of light emanating from a scattering layer, when the latter is lighted on one side by a parallel beam of light and is bounded on the other side by a plane wall reflecting the light diffusely. The albedo A_ν is given. The notation is as follows: $J_\nu(\tau)=(4\pi)^{-1}\int I_\nu(\tau, r')d\omega'$; I_ν is the intensity of light, $d\omega'$ the element of solid angle having the direction r' for axis, τ the distance from the reflecting wall; $0<\tau\leqq q$. The problem is solved precisely and uniquely on the basis of two integral equations of the form

$$J_\nu^{(i)}(\tau)=f_i(\tau)+\tfrac{1}{2}\int_0^q J_\nu^{(i)}(t)E_1(|\tau-t|)dt, \qquad i=1, 2;\ f_1=f,\ f_2=E_3,$$

defining $J_\nu^{(i)}(\tau)$. It is established that

$$J_\nu(\tau)=J_\nu^{(1)}(\tau)+c_\nu J_\nu^{(2)}(\tau),$$

where

$$c_\nu=a_\nu/b_\nu, \quad a_\nu=A_\nu\textstyle\int_0^q J_\nu^{(1)}E_2dt, \quad b_\nu=1-A_\nu\int_0^q J_\nu^{(1)}E_2dt.$$

An explicit formula is given for I_ν in terms of J_ν. The author also carries out, for the problem involved, an approximation method of the type usual in astrophysics.

W. J. Trjitzinsky (Urbana, Ill.).

Kuznetzov, E. S. Contribution to the problem of calculation of the field of radiation in an absorbing and scattering atmosphere for a given temperature distribution. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 241–246 (1942). [MF 7620]

The author is concerned with the integrodifferential equation of radiation transfer

$$\text{(1)} \qquad \frac{\cos\theta}{\rho(z)}\frac{\partial}{\partial z}I_\nu(z, r)=\eta_0(z)+\frac{\sigma_\nu(z)}{4\pi}\int I_\nu(z, r')\gamma_\nu(z; r', r)d\omega' -[\alpha_\nu(z)+\sigma_\nu(z)]I_\nu(z, r),$$

integration over unit sphere. Here $I_\nu(z, r)$ is radiation intensity (frequency ν); $\eta_\nu(z)$, $\sigma_\nu(z)$, $\alpha_\nu(z)$ are radiation, scattering and absorption coefficients, respectively; γ_ν describes the law of scattering; $\rho(z)$ is the density of the medium, z the altitude above the earth surface, r the direction of the ray, $d\omega'$ the surface element (connected with direction r') of unit sphere, θ the angle between the ray and the vertical. The unknown is $I_\nu(z, r)$. By a converging process of successive approximations the author extends the solution of (1), previously [Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1940**, 813–842; these Rev. **3**, 157] given by him for the case $\gamma_\nu=1$ and some symmetry cases, to the case of arbitrary scattering laws.

W. J. Trjitzinsky (Urbana, Ill.).

Functional Analysis

Schatten, Robert. On the direct product of Banach spaces. Trans. Amer. Math. Soc. **53**, 195–217 (1943). [MF 8119]

The direct product of two spaces E_1 and E_2, $E_1\otimes E_2$, is obtained by using the distributive symbol $\otimes$ and $+$ to set up elements or "expressions," $f_1\otimes g_1+\cdots+f_n\otimes g_n$, $f_i\in E_1$, $g_i\in E_2$, and completing by means of a suitable norm, which however is not uniquely determined. Two "expressions" are considered equal if one can be transformed into the other by means of the operations $(af)\otimes g=f\otimes(ag)$, $(f_1+f_2)\otimes g=f_1\otimes g+f_2\otimes g$. Algebraically these expressions behave like matrices. A norm for expressions is called a crossnorm if $N(f\otimes g)=\|f\|\cdot\|g\|$. Such a norm also has an associate norm $\bar{N}$, defined for expressions from $\bar{E}_1\otimes\bar{E}_2$. There is a greatest

crossnorm, and a least crossnorm whose associate is also a crossnorm. Beginning with these two, one may alternately average two and take the associate to obtain a sequence of crossnorms which converge to a crossnorm N. This latter norm is defined for every pair of Banach spaces and when applied to Hilbert spaces gives the usual direct product of two Hilbert spaces which is again a Hilbert space. [The direct product of two Hilbert spaces was defined in a paper by J. von Neumann and the reviewer, Ann. of Math. (2) 37, 116–229 (1936).] *F. J. Murray* (New York, N. Y.).

Wassilkoff, D. Partially ordered linear systems, Banach spaces and systems of functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 135–137 (1942). [MF 7653]

Let E be a partially ordered linear system such that $Tx=x+z$ and $Tx=\lambda x$, $\lambda>0$, are order-preserving, and such that for each pair $x, y\in E$ there is $z\in E$ with $z\geqq x$, $z\geqq y$. The following axioms are considered (omitting one): (A) if, for $x\in E$, $R_x=\{\lambda x \mid 0\leqq\lambda<\infty\}$ is bounded above, then $x\leqq 0$; (B^o) E contains an element $u>0$ such that each $x\in E$ can be represented as $x=\lambda u-x'$, where $\lambda\geqq 0$, $x'\in E^+$; (B) E is a Banach space whose unit sphere has an upper bound; (C) each set $M\subset E$ which has an upper bound has a least upper bound. If E satisfies, say, axioms (A) and (C) we shall say E is $E_{A,C}$. Let F be an arbitrary linear system of real functions defined on an abstract set Q with the ordering $f_1<f_2$ if $f(q)\leqq f(q)$ for each $q\in Q$ and $f_1(q)<f_2(q)$ for some q. Theorem 1: F is F_A. Conversely, if E is E_A, then E has a representation as a linear system of real functions on some abstract set Q with ordering as above. Theorem 2: There exists an extension E' of E such that E' is E'_C if and only if E is E_A. If an extension exists, there exists one which is minimal.

If E is E_{B^o}, $y\in E$, $\alpha>0$, the set of $x\in E$ such that $-(\alpha-\epsilon)u\leqq y-x\leqq(\alpha-\epsilon)u$ for some $\epsilon>0$ will be called an interval $U_\alpha(y)$. Theorem 3: Let E be E_{B^o}. In order that a topology can be defined in E such that (1) E becomes a T_1-space, (2) the set of all intervals is a basis for E, (3) algebraic operations in E are continuous, it is necessary and sufficient that E does not contain any bounded linear subsystem (for example, if E is E_{A,B^o}). If the condition is satisfied, the topology may be introduced by a norm, in particular, the u-norm: $\|x\|_u=\inf\{\lambda \mid -\lambda u\leqq x\leqq\lambda u\}$. Theorem 4: If E is $E_{B^o,C}$ and has a u-norm, E is complete in this norm. Let E be E_B, and call the given norm an N-norm. Theorem 5: E_B is $E_{A,B}$ if and only if E^+ is N-closed. Theorem 6: Let E be E_B and satisfy theorem 3. The N-norm and the u-norm define the same topology if and only if all intervals in E are N-bounded. No proofs are given. *J. V. Wehausen.*

Šmulian, V. Approximation in the space of bounded functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 242–248 (1942). [MF 7607]

The fundamental theorem of this note concerns (1) a general set Q and a particular element q_0 of Q, (2) a convex subset $\mathfrak{M}$ of the set $E(\theta)$ of all real valued functions on Q which satisfies the two conditions: (a) if $x_t(q)$ is any transfinite sequence from $\mathfrak{M}$, then there exists an element $x_0(q)$ of $\mathfrak{M}$ such that $\underline{\lim}_t x_t(q)\leqq x_0(q)\leqq\overline{\lim}_t x_t(q)$; (b) if, for any two functions of $\mathfrak{M}$, $x_1(q)=x_2(q)$ for all q except q_0, then $x_1(q_0)=x_2(q_0)$. Under these conditions, for every $\epsilon>0$ there exist constants $c_0, c_1, \cdots, c_n$ and elements $q_1, \cdots, q_n$ such that

$$|x(q_0)-c_0-\sum_{i=1}^{n} c_i x(q_i)|\leqq\epsilon$$

for all x of Q. If one considers, for fixed q, $x(q)$ as being a function ψ_q on $\mathfrak{M}$ the conditions on $\mathfrak{M}$ guarantee that ψ_q is a bounded function on $\mathfrak{M}$, and the conclusion asserts that ψ_{q_0} can be uniformly approximated by linear combinations of 1 and the functions ψ_q, $q\neq q_0$. The proof then reduces to showing that, if, for an additive function of subsets of $\mathfrak{M}$, $\varphi(e)$, and $q\neq q_0$: $\int_{\mathfrak{M}} x(q)d\varphi(e)=0$ and $\int_{\mathfrak{M}} 1\cdot d\varphi(e)=0$, then $\int_{\mathfrak{M}} x(q_0)d\varphi(e)=0$. Additional conditions on $\mathfrak{M}$ insure special properties of the constants; for instance, if the condition $|x(q_0)|\leqq C\sup_{q\neq q_0}|x(q)|$ holds for x on $\mathfrak{M}$, then $c_0=0$ and $\sum_{i=1}^{n}|c_i|<C$. A consequence of the theorem is that, if $f_0(x)$ is a linear functional on the space $E_0(Q)$ of bounded functions on Q, which is transfinitely continuous on $E_0(Q)$ [for every transfinite sequence $x_t(q)$ for which $\|x_t\|\leqq 1$, there exists an x_0 such that $\|x_0\|\leqq 1$, $\underline{\lim}_t x_t(q)\leqq x_0(q)\leqq\overline{\lim}_t x_t(q)$ for all q, and $\underline{\lim}_t f_0(x_t)\leqq f_0(x_0)\leqq\overline{\lim}_t f_0(x_t)$], then $f_0(x)$ is of the form $\sum_{i=1}^{n} c_i x(q_i)$ with $\|f_0\|=\sum_{i=1}^{n}|c_i|$.

T. H. Hildebrandt (Ann Arbor, Mich.).

Rickart, C. E. Integration in a convex linear topological space. Trans. Amer. Math. Soc. **52**, 498–521 (1942). [MF 7458]

This paper treats an integration process for multi-valued set functions which are defined over a σ-field $\mathfrak{M}$ and whose values lie in the convex linear topological space $\mathfrak{X}$ of von Neumann [Trans. Amer. Math. Soc. **37**, 1–20 (1935)]. The integral is a generalization of one treated by Kolmogoroff [Math. Ann. **103**, 654–696 (1930)], who considered only the case in which $\mathfrak{X}$ is the real numbers. The method of defining the integral is a generalization of that used by G. Birkhoff [Ann. of Math. (2) **38**, 39–56 (1937)] and Phillips [Trans. Amer. Math. Soc. **47**, 114–145 (1940); these Rev. **2**, 103]; this definition bases unconditional convergence on a Moore-Smith limit [T. H. Hildebrandt, Bull. Amer. Math. Soc. **46**, 957–962 (1940); these Rev. **2**, 221] in such a way that infinite sums are avoided. Convergence, in general that of a sequence of sets in $\mathfrak{X}$ to a similar set, is defined in terms of a generalization, applicable in a linear topological space, of the Hausdorff distance for sets in a metric space used by Price [Trans. Amer. Math. Soc. **47**, 1–50 (1940); these Rev. **1**, 239].

The paper is divided into four parts. Part I contains preliminary results on convex linear topological spaces, unconditional summability, and additive set functions. Part II contains the general theory of the V- and SV-integrals. The V-integral is multi-valued and is defined for multi-valued set functions $F(\sigma)$. If the set which is the value of the integral contains only a single point, the integral is called the SV-integral. The author treats the basic properties of the V- and SV-integrals, generalizes Kolmogoroff's notion of differential equivalence and applies it to the V-integral, shows that the transform of an integrable function by a general type of linear transformation is integrable, and treats a convergence theorem for the SV-integral which involves a generalization of the notion of approximate convergence for functions $F(\sigma)$. Part III treats the V_B-integral; it involves integration with respect to a bilinear function $B[y,\sigma]$ which generalizes $m(\sigma)y$, where $m(\sigma)$ is a numerical measure function. $B[y,\sigma]$ has its values in $\mathfrak{X}$ and is defined for y in a linear space $\mathfrak{Y}$ and σ in the σ-field $\mathfrak{M}$. It is linear in y for each σ and completely additive in σ for each y. Functions $y(\sigma)$ to be integrated have their values in $\mathfrak{Y}$; the value of the integral is in $\mathfrak{X}$. Part IV compares the integrals treated in this paper with others previously defined. The

SV-integral, when $\mathfrak{X}$ is the real numbers, includes an integral of Kolmogoroff. The V_B-integral reduces in a special case to an integral of Phillips, and a specialization of the V_B-integral includes an integral of Price. Relations of the V_B-integral to various other integrals can be obtained through its relation to the Phillips integral.

G. B. Price (Lawrence, Kan.).

Julia, Gaston. Sur les projecteurs de l'espace hilbertien ou unitaire. C. R. Acad. Sci. Paris **214**, 456–458 (1942). [MF 7858]

The author considers two projections P_1, P_2 of a Hilbert or unitary space $\mathfrak{H}$ onto two linear closed manifolds $\mathfrak{M}_1$, $\mathfrak{M}_2$ and the lincar closed manifolds $\mathfrak{N}_1=\mathfrak{M}_1\theta(\mathfrak{M}_1\cdot\mathfrak{M}_2)$, $\mathfrak{N}_2=\mathfrak{M}_2\theta(\mathfrak{M}_1\cdot\mathfrak{M}_2)$. The first theorem proved is that P_1 and P_2 are permutable if and only if the manifolds $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are orthogonal. It follows from this result that $P_1=P_1P_2+P_1(I-P_2)$, $P_2=P_2P_1+P_2(I-P_1)$ and hence every pair P_1, P_2 of permutable projections can be expressed in the form $P_1=A_1+A_2$, $P_2=A_1+A_3$, where A_1, A_2, A_3 are the projections of $\mathfrak{H}$ onto three mutually orthogonal manifolds $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$. The paper essentially closes with the following result: if P_1 and P_2 are projections determined by the linear closed manifolds $\mathfrak{M}_1$, $\mathfrak{M}_2$ and if P is a projection permutable with P_1, P_2, then P is permutable with the projection determined by $\mathfrak{M}_1+\mathfrak{M}_2$, the linear manifold determined by the logical sum $\mathfrak{M}_1+\mathfrak{M}_2$. This theorem seems false since $\mathfrak{M}_1+\mathfrak{M}_2$ is not necessarily closed and hence does not necessarily determine a projection; moreover the proof given in the paper fails if $\mathfrak{M}_1+\mathfrak{M}_2$ is replaced by $\mathfrak{M}_1\oplus\mathfrak{M}_2$, the linear closed extension of $\mathfrak{M}_1$ and $\mathfrak{M}_2$. *H. H. Goldstine.*

Julia, Gaston. Sur la représentation analytique des opérateurs linéaires dans l'espace hilbertien. C. R. Acad. Sci. Paris **214**, 591–593 (1942). [MF 7890]

The author showed in previous notes [same C. R. **212**, 733–736, 829–831 (1941); these Rev. **3**, 51, 208] that every bounded linear operator A of a Hilbert space $\mathfrak{H}$ could be represented in the form $AX=\sum(\phi_i, X)A\phi_i$, the series converging strongly for every vector X and the ϕ_i forming a complete orthonormal base in $\mathfrak{H}$. In the present paper linear closed operators A, whose domain determines $\mathfrak{H}$, are considered and it is indicated how A is representable in the form $AX=\sum_i\sum_k(\phi_{ik}, X)A\phi_{ik}$. The ϕ_{ik} form in the domain of A a complete orthonormal system for $\mathfrak{H}$ and are so selected that $A\phi_{ik}$ is orthogonal to $A\phi_{jk}$ for $i\neq j$. If $\mathfrak{M}_i$ is the closed linear manifold determined by the set $(\phi_{ik}|k=1, 2, \cdots)$, then it is shown that the sum $c_i=\sum_k(\phi_{ik}, X)A\phi_{ik}$ converges strongly in $\mathfrak{M}_i$ and $AX=\sum c_i$ converges strongly in the domain of the operator A.

H. H. Goldstine (Philadelphia, Pa.).

Julia, Gaston. Représentation paramétrique bornée des opérateurs linéaires non bornés de l'espace hilbertien. C. R. Acad. Sci. Paris **214**, 709–710 (1942). [MF 7886]

It is shown that every linear closed operator $y=Ax$ in a Hilbert space $\mathfrak{H}$ is capable of a parametric representation in the form $x=Kz$, $y=K_1z$, where K and K_1 are linear bounded operators defined over $\mathfrak{H}$. The representation is accomplished by expressing A as BH, where H is self-adjoint and B is a bounded operator that is isometric in the domain of H. The operator K is the inverse of $H-\lambda$, with λ imaginary, and $K_1=B+\lambda BK$. *H. H. Goldstine.*

Hille, Einar. On the analytical theory of semi-groups. Proc. Nat. Acad. Sci. U. S. A. **28**, 421–424 (1942). [MF 7289]

Continuing his studies [Proc. Nat. Acad. Sci. U. S. A. **28**, 175–178 (1942); these Rev. **4**, 13] of a semi-group T_s, $s>0$, of linear operators in a complex Banach space E, the author improves one of his results and also discusses the behavior of T_h for small h. Let $D(A)$ be the set of y for which $Ay=\lim_h A_hy$ exists, where $A_h=h^{-1}[T_h-I]$. If $\sup_{0<s<\omega}|T_s|=M(\omega)<\infty$ and, for some particular x, $T_hx\to x$ and $\liminf_h|A_hx|=0$, then $T_sx=x$ for all s. If $T_sx\to x$ for all x, then $|A_hy|\leqq M(h)|Ay|$ for y in $D(A)$. If $|T_s|\leqq 1$ for $s>0$ and $T_sx\to x$ on E, then $|\exp(sA_h)|\leqq 1$, $s\geqq 0$, $h>0$, and $\exp(sA_h)x\to T_sx$ uniformly for $0\leqq s\leqq\omega<\infty$. The first of these theorems is applied to Abel-Poisson summability of Fourier series and the second to the space of continuous functions, where an analogue of Taylor's series is obtained. Further results relating the spectral properties of A and those of T_s are given. *N. Dunford* (New Haven, Conn.).

Theoretical Statistics

Menger, Karl. Statistical metrics. Proc. Nat. Acad. Sci. U. S. A. **28**, 535–537 (1942). [MF 7642]

A statistical metric is defined by the author as a set S such that with each two elements (points) p and q of S a probability distribution function $\pi(x; p, q)$ is associated satisfying the following conditions: (1) $\pi(0; p, q)=1$. (2) If $p\neq q$, then $\pi(0; p, q)<1$. (3) $\pi(x; p, q)=\pi(x; q, p)$. (4) $T[\pi(x; p, q), \pi(y; q, r)]\leqq\pi(x+y; p, r)$, where $T(\alpha, \beta)$ is a function, defined for $0\leqq\alpha\leqq 1$ and $0\leqq\beta\leqq 1$, satisfying certain conditions. The function $\pi(x; p, q)$, called distance function, is interpreted as the probability that p and q have a distance not greater than x. The author shows that large parts of metric geometry can be developed in statistical metric spaces, in particular, a theory of betweenness. Applications of statistical metrics to macroscopic, microscopic and physiological spatial measurements are briefly discussed. It is pointed out that statistical metrics may provide us with methods removing some conceptual difficulties and paradoxes of the classical theory. *A. Wald.*

Curtiss, J. H. A note on the theory of moment generating functions. Ann. Math. Statistics **13**, 430–433 (1942). [MF 7873]

Let X be a random variable and $F(x)$ its cumulative distribution function. Then $G(\alpha)=E(e^{\alpha x})=\int_{-\infty}^{+\infty}e^{\alpha x}dF(x)$ (α real) is called the moment generating function of X, provided the integral exists and is finite for values of α in a finite interval containing the origin. The moment generating function is closely related to the characteristic function given by $E(e^{itx})$ where t is real and $i=\sqrt{-1}$. It is known that a characteristic function uniquely determines the distribution function, and that, if a sequence of characteristic functions converges to a limit, the corresponding sequence of distribution functions does likewise. The author shows in this note that to a certain extent analogous theorems may be stated for moment generating functions. *A. Wald.*

Satterthwaite, F. E. Generalized Poisson distribution. Ann. Math. Statistics **13**, 410–417 (1942). [MF 7870]

By a generalized Poisson distribution the author understands distribution functions of the form

$$F(x, t)=e^{-\lambda t}\sum_{k=0}^{\infty}(\lambda t)^k\phi_k(x)/k!,$$

where λ and t are parameters and $\phi_k(x)$ the kth iterated convolution of a distribution function with itself. This actually means a rediscovery of the general form of the doubly-homogeneous discontinuous stochastic process of Markoff type [cf., for example, A. Khintchine, Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Ergebnisse der Mathematik, vol. 2_4, Springer, Berlin, 1933, in particular, chap. 2, § 4]. An application to sickness and accident insurance is given. The findings may be compared with those of O. Lundberg [On Random Processes and Their Application to Sickness and Accident Statistics, Uppsala, 1940; these Rev. **2**, 230]; the methods of approach and the point of departure are quite different, however.

W. Feller (Providence, R. I.).

Girshick, M. A. A correction. Ann. Math. Statistics **13**, 447 (1942). [MF 7876]

Concerning the author's paper in the same Ann. **13**, 235–238 (1942); these Rev. **4**, 21.

Dwyer, Paul S. Recent developments in correlation technique. J. Amer. Statist. Assoc. **37**, 441–460 (1942). [MF 7535]

This is the fifth of a series of papers in which the author has described, and improved, the modern computational methods for linear equations [Psychometrika **6**, 101–129, 191–204, 355–365 (1941); these Rev. **2**, 367; **3**, 154; Ann. Math. Statistics **12**, 449–458 (1941); these Rev. **3**, 276]. The simplification is due to the use of a compact method, eliminating unnecessary entries. In the present paper the author reviews the applications of his method to the computational correlation technique. All entries in the author's method of solving normal equations are immediately interpretable as multiple correlation and regression constants.

W. Feller (Providence, R. I.).

Wei, Dzung-shu. Necessary and sufficient conditions that regression systems of sums with elements in common be linear. Nat. Math. Mag. **17**, 151–158 (1943). [MF 7774]

Suppose that y and z are linear forms in $k+n$ and $k+m$ random variables all obeying the same distribution law and that the first k variables in the one form are identical with the first k in the other. If the coefficients in y are all equal, it is known that the regression of z on y is linear. In this paper the necessary and sufficient condition for arbitrary coefficients that this regression be linear is obtained with some discussion of the result. *C. C. Craig.*

Vali, M. A. On the sampling distribution of harmonic means. Bull. Calcutta Math. Soc. **34**, 87–91 (1942). [MF 7533]

The author derives the distribution of the harmonic mean of a random sample for normal and Pearson's type III distributions. The results are obtained with the help of the characteristic function of the reciprocal of the variate x under consideration. It seems to the reviewer that the deductions are not correct. That the final results are not valid can be seen from the fact that, for samples of size one, the harmonic mean H is equal to the variate x but the distribution of H is not equal to that of x according to the formula given by the author. *A. Wald.*

Nair, A. N. Krishnan. Distribution of Student's "t" and the correlation coefficient in samples from non-normal populations. Sankhyā **5**, 383–400 (1941). [MF 7356]

The question of the distribution of Student's t in samples from nonnormal populations has been extensively discussed in the literature. Theoretical as well as experimental investigations have been made. The latter consist in drawing random samples from a nonnormal population and comparing the observed distribution of t with its theoretical distribution in samples from a normal population. In the present paper the experimental method is used. From the population A following the probability law e^{-x}, 865 random samples of 6 were drawn, and 518 random samples of 5 from the population B following the elementary probability law cxe^{-x}. The author finds that the agreement between the observed distribution of t and the corresponding "Student's" distribution is not very close, which is not very surprising in view of the small sample sizes (5 and 6) used in this investigation. The author also calculates 433 sample values of the correlation coefficient r from population A and 260 from population B, and compares the observed distributions with the corresponding theoretical distribution of r valid for normal populations. He finds that for population B there is a close agreement, while for population A the agreement is slightly upset in the range .85 to .95. *A. Wald.*

Nair, U. Sivaraman. Probability statements regarding the ratio of standard deviations and correlation coefficient in a bivariate normal population. Sankhyā **5**, 151–156 (1941). [MF 7350]

The author derives confidence limits for the ratio of the standard deviations and the correlation coefficient in a bivariate normal population, assuming that one of these two parameters is known. The problems treated here are similar to those investigated by S. S. Bose, D. J. Finney and W. A. Morgan. However, the methods used in this paper and the results obtained are in some respects different.

A. Wald (New York, N. Y.).

de Varennes e Mendonça, P. Orthogonality and analysis of variance. Portugaliae Math. **3**, 234–252 (1942). [MF 7978]

This is an exposition of the mathematics of Fisher's analysis of variance for a randomized block employing some notation and terminology due to the author. The statement, not essential in his development, that the maximum likelihood estimate of the population variance is also unbiased is known not to be true, at least for a normal universe.

C. C. Craig (Ann Arbor, Mich.).

Scheffé, Henry. On the ratio of the variances of two normal populations. Ann. Math. Statistics **13**, 371–388 (1942). [MF 7867]

Given two random samples from a normal population with unknown means and variances, this paper is concerned both with significance tests for the hypothesis that the ratio θ of the unknown population variances has a given value θ_0 and with the determination of confidence intervals for θ. In the first part, which is less technical mathematically, six significance tests based on the use of one or both tails of the F-distribution are discussed, with consideration also of the associated confidence intervals. The two-sided tests also include, besides the more usual reciprocal limits and equal limits, tests based on the likelihood ratio and on the condition that the corresponding confidence intervals be

logarithmically shortest. By consideration of the power functions it appears that the last of these four is to be preferred. In the case of unequal sample sizes the limits to use in this test are numerically difficult to compute but, in the case that both samples contain ten or more items, evidence is presented to show that the use of the easily obtained equal tails limits involves little loss in efficiency. In the second part, it is shown first that the one-sided tests are the best possible not only if (as is known) the alternatives for θ are limited to $\theta<\theta_0=1$ or $\theta>\theta_0=1$ but also for the case when the alternatives are simply $\theta<\theta_0$ or $\theta>\theta_0$. Then, since no uniformly most powerful test exists, tests unbiased in the Neyman-Pearson sense are considered and the preferred test of the first section is shown not only to be of Neyman's type B but also of type B_1 as defined by the present author [Ann. Math. Statistics **13**, 280–293 (1942); these Rev. **4**, 127]. The associated confidence intervals may then be called shortest unbiased.

C. C. Craig (Ann Arbor, Mich.).

Hart, B. I. Significance levels for the ratio of the mean square successive difference to the variance. Ann. Math. Statistics **13**, 445–447 (1942). [MF 7875]

Let $x_1, x_2, \cdots, x_n$ be the values of a random variable x in a sample of size n from a normal population. Let

$$\delta^2=(n-1)^{-1}\sum_1^{n-1}(x_{i+1}-x_i)^2,$$

and $s^2=n^{-1}\sum_1^n(x_i-\bar{x})^2$, where $\bar{x}$ is the sample mean, and let $u=\delta^2/s^2$. von Neumann [Ann. Math. Statistics **12**, 367–395 (1941); these Rev. **4**, 21] has determined the probability element $w(u)d(u)$ of u. In the present paper Miss Hart has tabulated 0.1, 1, 5, 95, 99, 99.1 percent significance levels of u for values of n from 4 to 60. More precisely, she has tabulated the values of k for which

$$\int_0^k w(u)du=.001, .01, .05$$

for $n=4$ to $n=60$, and the values of k' for which

$$\int_{k'}^\infty w(u)du=.999, .99, .95$$

for $n=4$ to $n=60$. *S. S. Wilks* (Princeton, N. J.).

Guilford, J. P. and Lyons, Thoburn C. On determining the reliability and significance of a tetrachoric coefficient of correlation. Psychometrika **7**, 243–246 (1942). [MF 7566]

In this paper the authors give some tables which facilitate the calculation of the standard deviation of the tetrachoric correlation r. In further tables significant values of r for different sample sizes are given on the 5% and 1% levels.

A. Wald (New York, N. Y.).

Nair, K. R. Table of confidence interval for the median in samples from any continuous population. Sankhyā **4**, 551–558 (1940). [MF 7360]

W. R. Thompson and S. R. Savur have independently obtained confidence intervals for the median when nothing is known about the population distribution except that it is continuous. In this paper a discussion of these results is given and a table of confidence coefficients for different sample sizes is prepared. The table is based on Thompson's results. *A. Wald* (New York, N. Y.).

Chandra Sekar, C. and Francis, Mary G. A method to get the significance limit of a type of test criteria. Sankhyā **5**, 165–168 (1941). [MF 7352]

The author considers test criteria based on the maximum of a set of statistics. In particular, the distribution of the maximum M of the k statistics $M_1, \cdots, M_k$ is studied, where $M_i=m_i(m_1+\cdots+m_k)^{-1}$ and $m_1, \cdots, m_k$ are mutually independent, each having the χ^2 distribution with f degrees of freedom. It is shown that, in the interval $[\frac{1}{k}, 1]$, the probability density function of M is equal to k times the probability density function of M_i. Since the distribution of M_i is relatively simple, one can easily obtain the significance point of M, provided it lies in the interval $[\frac{1}{k}, 1]$. These results are closely related to some more general results of W. G. Cochran [Ann. Eugenics **11**, 47–52 (1941); these Rev. **3**, 171] which appeared at approximately the same time as this paper. *A. Wald* (New York, N. Y.).

Geary, R. C. The estimation of many parameters. J. Roy. Statist. Soc. (N.S.) **105**, 213–217 (1942). [MF 8014]

If $\|c_{ij}\|$ $(i, j=1, 2, \cdots, h)$ is the matrix of variances and covariances for h random variables, then the determinant $|c_{ij}|$ is known as the generalized variance of the random variables. Let π be a multinomial population in which there are k mutually exclusive classes $C_1, C_2, \cdots, C_k$ with probabilities $p_1, p_2, \cdots, p_k$ which are functions of h parameters $\theta_1, \theta_2, \cdots, \theta_h$ satisfying certain regularity conditions [given in general form by Wilks and Daly, Ann. Math. Statistics **10**, 225–235 (1939); these Rev. **1**, 64]. The author shows that for large samples the generalized variance of the maximum likelihood estimates of the θ's is smaller than that for any other set of estimates of the θ's (satisfying certain regularity conditions). This result is closely related to the general results of Wilks and Daly [loc. cit.].

S. S. Wilks (Princeton, N. J.).

Wilks, S. S. Statistical prediction with special reference to the problem of tolerance limits. Ann. Math. Statistics **13**, 400–409 (1942). [MF 7869]

Let us assume that X_1 and X_n are the smallest and largest values in a sample of n of a continuous random variable X. The following problems of prediction relating to a second sample of N are considered: (1) What is the probability P' that at least N_0 values in the second sample will exceed the tolerance limit X_1 set by the first sample? (2) What is the probability P that at least N_0 values of X in the second sample will lie between the two tolerance limits X_1 and X_n set by the first sample? (3) For given values of n, N_1, and α, what is the largest integer N_α such that the probability P' is at least α that $N_0 \leqq N_\alpha$? What is the limiting value of $N_\alpha/N=\bar{R}_\alpha$ as $N\to\infty$? Tables of values of N_α and $\bar{R}_\alpha$ for several important pairs of values of n and N are given for $\alpha=.99$ and .95. Problems similar to (1), (2), and (3) are discussed for the case where tolerance limits are simultaneously placed on more than one quality characteristic. Comments particularly valuable to the applied statistician are made on the generality of the theory of tolerance limits and how it applies to differences, products, ratios, and other functions of more than one random variable.

W. A. Shewhart (New York, N. Y.).

Wald, Abraham. Setting of tolerance limits when the sample is large. Ann. Math. Statistics **13**, 389–399 (1942). [MF 7868]

The author assumes that a sample of n has been chosen from a p-variate population involving k unknown param-

eters. He considers the important practical problem of finding the p pairs U_i, L_i $(i=1, 2, \cdots, p)$ of most favorable tolerance limits to be applied to a second sample of N from the same population in that U_i-L_i is to be a minimum in some sense. Two problems are considered, viz., $N=\infty$, and N is finite. Solutions of these two problems are given for large sample size n. Wilks [in the paper reviewed above] has given solutions for the univariate case, i.e., where $p=1$. Wilks' solution appears to be the best possible one if nothing is known about the probability density function except that it is continuous. However, if it is known a priori that the unknown density function is an element of a k-parameter family of functions, it will in general be possible to derive tolerance limits that are better in a certain sense than those derived by Wilks on the more general assumption. The two problems are considered for both the univariate and multivariate cases. Solutions are sketched for the example of a normally distributed variate. No numerical examples are given and the practical question of how large n must be in order to be considered large is not discussed.

W. A. Shewhart (New York, N. Y.).

Gumbel, E. J. Statistical control-curves for flood-discharges. Trans. Amer. Geophys. Union **1942**, 489–509 (1942). [MF 7907]

Haldane, J. B. S. Selection against heterozygosis in man. Ann. Eugenics **11**, 333–340 (1942). [MF 7972]

GEOMETRY

Everett, C. J. Affine geometry of vector spaces over rings. Duke Math. J. **9**, 873–878 (1942). [MF 7936]

The principal result of this paper is theorem 5: A geometry satisfying axioms I–VII of Artin [Rep. Math. Colloquium (2) **2**, 15–20 (1940); these Rev. **3**, 179] admits a coordinate system in which $P=(\alpha, \beta)$, with α, β in a ring $\mathfrak{R}$ with unit, without zero-divisors, and with every pair of elements not both zero possessing a "greatest" common right divisor. The points on a line have the form $(\alpha+\mu\kappa, \beta+\nu\kappa)$, where α, β are fixed, where μ, ν are fixed and not both 0 and have g.c.d. 1 and where κ is arbitrary in $\mathfrak{R}$.

C. C. Torrance (Cleveland, Ohio).

Hjelmslev, Johannes. La géométrie sensible. II. Enseignement Math. **38**, 294–322 (1942). [MF 7305]

[The first part appeared in the same volume, 7–26 (1940); these Rev. **1**, 322.]

The author develops elementary geometry in such a way that the deviations of "practical" geometry from "ideal" geometry are taken into account. Assume that the practical exactness is within ± 1 mm. Then a piece of graph paper with 1 mm spacing is used of size 1 qm, say. With the center as origin we obtain the piece $|x|<500$, $|y|<500$ of the coordinate plane, practically significant points being the points with integral coordinates. The main idea is that large parts of elementary analytic geometry and trigonometry can be derived by making use of these points only. Then the results will be consistent with the following facts: the lines $y=x/200$ and $y=0$ coincide for 16 mm; a circle of radius 200 mm has a common intersection of 4 mm with its tangent; two lines do not intersect as soon as their ordinary intersection lies outside of the square $|x|<500$, $|y|<500$, etc.

H. Busemann (Chicago, Ill.).

Terracini, Alejandro. On some geometrical loci. Union Mat. Argentina, Publ. No. **22**, 11 pp. (1941). (Spanish) [MF 7951]

This paper deals with the following problem. Let the points A and B run over the straight lines a and b, respectively. For every pair A, B, the point P shall be chosen in such a way that ABP is (directly) similar to a given triangle $A_0B_0P_0$. To find all those triangles $A_0B_0P_0$ and pairs of straight lines a, b, for which the set of all the points P degenerates into a straight line.

P. Scherk.

Cobb, R. H. Some homothetic triangles related to the Euler line. Math. Gaz. **26**, 209–211 (1942). [MF 7977]

Gonzalez, Mario O. Generalization of Menelaus' theorem. Revista Ci., Lima **44**, 93–106 (1942). (Spanish) [MF 7015]

The trigonometric form of Menelaus' theorem for the triangle is extended to a polygon of any number of sides. Three proofs are given. The last proof uses polar reciprocation and validates the proposition for the case when the transversal is the line at infinity. The segmentary form of Menelaus theorem was generalized by L. N. M. Carnot [Géométrie de position, Paris, 1803, p. 295, art. 241].

N. A. Court (Norman, Okla.).

Mächler, W. On a property of real conic sections with center. Revista Union Mat. Argentina **8**, 145–149 (1942). (Spanish) [MF 8025]

Yates, R. C. Folding the conics. Amer. Math. Monthly **50**, 228–230 (1943). [MF 8213]

Iyengar, K. V. A note on Poncelet's problem. J. Mysore Univ. Sect. B. **3**, 131–134 (1942). [MF 7797]

By a simple geometric method, there is obtained a recurrence formula for computing λ_m such that P_1P_m touches the conic $S_1+\lambda_mS_2$, where $P_1, P_2, P_3, \cdots$ are points on a conic S_1 and such that P_rP_{r+1} $(r=1, 2, 3, \cdots)$ touches another conic S_2. Examples are given to illustrate the use of the formula.

J. L. Dorroh (Baton Rouge, La.).

Gaspar, Eduardo. The curve of contact of a conoid with its directrix. Revista Union Mat. Argentina **8**, 126–130 (1942). (Spanish) [MF 7676]

Rey Pastor, J. Spherical conoid with two rectilinear directrices. Revista Union Mat. Argentina **8**, 131–135 (1942). (Spanish) [MF 7677]

Simple considerations in connection with the following problem: given a sphere S and a straight line L outside of S. The tangents to S which meet L and are perpendicular to L form a "spherical conoid." To construct the tangents to the curve of contact of this conoid with S.

P. Scherk (Bloomington, Ind.).

Sutton, R. M. An instrument for drawing confocal ellipses and hyperbolas. Amer. Math. Monthly **50**, 253–254 (1943). [MF 8215]

Algebraic Geometry

Chang, Su-Cheng. The point of inflexion of a plane curve. Duke Math. J. **9**, 823–832 (1942). [MF 7932]

The neighborhoods of higher orders of an inflection O of a plane curve C are investigated by methods similar to those used by Bompiani but differing in the auxiliary curve employed, that is, Bompiani used cuspidal cubics and the author makes use of a bicuspidal quartic C_4 whose two cuspidal tangents and a stationary tangent are concurrent. Several new theorems are obtained involving configurations in the neighborhoods of O. *T. R. Hollcroft.*

Chang, Su-Cheng. The singularity S_1^m of a plane curve. Duke Math. J. **9**, 833–845 (1942). [MF 7933]

By a singularity S_1^m of a plane curve C, the author means a point of C at which the tangent line has contact of order m. The case $m=3$ has been treated by Bompiani and the two cases $m=3, 4$ by the author. Buchin Su [J. Chinese Math. Soc. **2**, 153–173 (1940); these Rev. **2**, 299] has generalized Bompiani's osculants to curves with representable singularities of higher orders. In the present paper, the author extends his own method of representing the neighborhoods of S_1^4 to S_1^m, $m>4$. He also obtains further properties of the neighborhoods studied by B. Su, a geometrical interpretation of the conditions for a representable S_1^m, additional covariant configurations associated with neighborhoods of high orders and a supplement to the canonical expansion of B. Su for two species of a representable S_1^m. *T. R. Hollcroft* (Aurora, N. Y.).

Johnson, R. P. Conical roulettes. Nat. Math. Mag. **17**, 202–211 (1943). [MF 8020]

The author considers two cones with common vertex and a common ruling. These cones will intersect in the common ruling and a cubic. If one cone is rolled onto a plane, the cubic becomes the conical roulette. The rolling is accomplished by passing a pencil of planes through the common ruling such that each plane of the pencil determines a point of the cubic. The parametric equations of the roulette are found. Various numerical examples are given and the curves of the corresponding roulettes are drawn. *N. Coburn* (Austin, Tex.).

Edge, W. L. Sylvester's unravelment of a ternary quartic. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 247–259 (1942). [MF 7961]

In an earlier paper [same Proc. **61**, 140–159 (1941); these Rev. **3**, 184] the author studied a plane quartic curve φ in (1, 1) correspondence with the curve of intersection of the Veronese surface F in [5] and a quadric primal, out polar to F. Covariants and contravariants of φ are represented by certain associated curves in [5]. In the present paper the method used in both papers makes clear that various results, including those of Clebsch [J. Reine Angew. Math. **69**, 125–145 (1861)], Ciani [Accad. Naz. Lincei. Rend. (5) 4_2, 274–280 (1855)] and Coble [Trans. Amer. Math. Soc. **4**, 65–85 (1903)], are all included in the unravelment of Sylvester [Cambridge and Dublin Math. J. **7**, 52–97 (1852) or Mathematical Papers, vol. I, pp. 284–327]. *V. Snyder.*

Gauthier, Luc. Au sujet d'un théorème de M. Apéry sur les quintiques. C. R. Acad. Sci. Paris **214**, 408–410 (1942). [MF 7854]

The plane quintic curve with five cusps is uniquely fixed by the position of the cusps on a proper conic and is self-dual. That the duality is a conic polarity was found by Apéry [C. R. Acad. Sci. Paris **213** (1941)] by transcendental methods. The present note derives this conic and discusses the possibility that it be composite. When it is, the quintic is composite. The locus of the fifth cusp consists of the sides of the quadrangle of the other four and of a curve, taken three times, from the points of which the lines to the other four cusps are equianharmonic. *V. Snyder* (Providence, R. I.).

Apéry, Roger. Sur un procédé de définition de courbes ayant un nombre élevé de rebroussements. C. R. Acad. Sci. Paris **214**, 46–47 (1942). [MF 7832]

Consider the correspondence defined by the three equations $x_i=f_i(x_1', x_2', x_3')$, in which each f_i is a form of degree m in the x', and the three curves $f_i=0$ have no common points. By this transformation a curve C in x of order h becomes a curve of order mh; each point is transformed into m points of the same nature, hence cusps into cusps. Each point of contact of C and the curve of branch-points of the transformation becomes a cusp on the Jacobian of the net $|f|$. If C of order h has r cusps and touches D in k points, then C' is of order mh and has $r'=m^2r+k$ cusps. Also $r'/m^2h^2 \geqq r/h^2$. By Plücker's numbers, r/h^2 has an unattainable upper bound. For $m=2$, D is any curve of class 3; hence C' is a curve of order 12 with 39 cusps. If D has a double tangent, C' is a C_8 with 15 cusps and one double tangent. Can this curve exist? *V. Snyder.*

Apéry, Roger. Sur la non-existence de courbes planes du huitième degré de genre 5 admettant $r \geqq 14$ rebroussements. C. R. Acad. Sci. Paris **214**, 340–341 (1942). [MF 7864]

Consider the given curve C_8, $p=5$, $k=14$, to be the projection of a space curve K from a point O not lying on it. A cusp on C_8 may arise from a cusp on K or from a tangent to K from O. The $|S_4|$ system of quartic surfaces through the proper cusps of K includes a subsystem containing K. The polar surfaces of this subsystem as to O form a system of cubics through the proper cusps; for a sufficiently high dimension, there is an S_4 through K, the polar cubic of which as to O also contains K. The residual intersection of S_4 and its polar cubic is a quartic with four apparent double points, hence composite. In this case K is also composite. By considering all possible cases, it follows that when K is a proper curve, $p=5$, it cannot have more than 13 cusps. The proof of Zariski [Amer. J. Math. **53**, 309–318 (1931)] that a C, $p=5$, with 16 cusps cannot exist is thus extended to the nonexistence of C, $p=5$, with 15 cusps and one double point and to C with 14 cusps and 2 double points. Similar restrictions can be applied to C with higher genera [cf. the paper reviewed above]. *V. Snyder* (Providence, R. I.).

Ramamurti, B. On ten associated points in [4]. Proc. Indian Acad. Sci., Sect. A. **16**, 191–192 (1942). [MF 7793]

The author gives a new proof of the following theorem: if ten points in [4], not lying on a rational quartic curve, are such that every quadric through nine of them passes through the tenth, then the rational quartic curve through any seven of them has the plane of the remaining three as a trisecant plane, and the three points and the three intersections with the quartic lie on a conic. *R. J. Walker.*

Segre, B. The postulation of a multiple curve. Proc. Cambridge Philos. Soc. **38**, 368–377 (1942). [MF 7804]

The object of the paper is to give a rigorous derivation of the postulation θ_h of a multiple h-fold curve C for hyper-

surfaces of a sufficiently high order. This object is attained by avoiding degeneration arguments which have been used in a previous derivation by Todd [Proc. Cambridge Philos. Soc. 36, 27–33 (1940); these Rev. 1, 168]. The proof is by induction with respect to the multiplicity k, since for $k=1$ the postulation formula is well established (Castelnuovo). The author considers then the hypersurfaces of a sufficiently high order which already go through C with multiplicity k, and he has only to find the number $\theta(\sigma)$ of additional independent conditions which are introduced by requiring that these hypersurfaces touch, along C, $k+1$ generic hypersurfaces through C. The proof that $\theta(\sigma)$ is actually the difference between the expected postulations θ_{k+1} and θ_k is again by induction, but this time the induction requires that the above conditions of contact be regarded as a special case of a more general set of contact conditions. Although the numerical computations become correspondingly more involved, the proof is essentially simple enough and makes use only of well-established results from the geometry of linear series on an algebraic curve and of linear systems on a surface. *O. Zariski* (Baltimore, Md.).

Martinelli, Enzo. Sulla imagine proiettiva delle serie e dei sistemi d'equivalenza elementari sopra una varietà. Pont. Acad. Sci. Acta 6, 147–151 (1942). [MF 7470]

A projective model of an elementary system of equivalence is obtained by considering the variety of Segre which represents the set of linear systems whose intersection determines the elementary system. The author examines the representation of elementary systems which are totally contained in the given one. *J. A. Todd.*

Bronowski, J. The fixed part of the canonical system. Proc. Cambridge Philos. Soc. 39, 31–34 (1943). [MF 7966]

An example is given to disprove the conjecture [P. DuVal, Proc. Cambridge Philos. Soc. 34, 1–5 (1938)] that there exist no surfaces on which the unassigned fixed part of the canonical system has a component which meets the variable part of the system in variable points and which cannot be transformed into a simple point by a birational transformation of the surface. The example is an octavic surface f having 4-fold points at seven of eight associated points and a tacnode at the eighth. Representations of f as a double Veronese surface and as a double plane are discussed.
R. J. Walker (Aberdeen, Md.).

Aprile, Giorgio. Il teorema di Bézout-Severi ed i sistemi algebrici ∞^d di S_k dell' S_r. Pont. Acad. Sci. Acta 6, 171–179 (1942). [MF 7469]

The author proves that any algebraic system Γ of S_k's in S_r can be reduced by continuity to a system Γ' which consists of the sum of systems satisfying an elementary Schubert condition. From this it follows that the Schubert varieties form an algebraic base on the corresponding Grassmannian variety. [This result was anticipated by Hodge; see J. London Math. Soc. 16, 245–255 (1941); these Rev. 3, 304.] *J. A. Todd* (Cambridge, England).

Goddard, L. S. Bases for the prime ideals associated with certain classes of algebraic varieties. Proc. Cambridge Philos. Soc. 39, 35–48 (1943). [MF 7967]

Let V_d be an algebraic variety given by a prime polynomial ideal p. If V_{d-1} is complete intersection of V_d with an hypersurface $f^*=0$, the ideal (p, f^*) does not necessarily coincide with the prime ideal p^* of V_{d-1} (it may differ from p^* by embedded components). It is proved that a necessary and sufficient condition that $p^*=(p, f)$ is that every hypersurface V^*_{d-1} containing V_{d-1} cuts V_d residually in a V_{d-1} which is also complete intersection. The author applies this result to various classes of varieties (such as varieties of Segre or Veronese) with the object of finding explicitly a base for their prime ideals and for the prime ideals of their subvarieties. *O. Zariski* (Baltimore, Md.).

Convex Domains, Integral Geometry, Extremal Problems

Pedoe, D. An inequality for two triangles. Proc. Cambridge Philos. Soc. 38, 397–398 (1942). [MF 7808]

A proof of the inequality

$$a'^2(-a^2+b^2+c^2)+b'^2(a^2-b^2+c^2)+c'^2(a^2+b^2-c^2)\geqq 16\Delta'\Delta,$$

connecting two triangles ABC and $A'B'C'$, by considering the circumcenters of triangles similar to $A'B'C'$ erected on the sides of ABC. *R. J. Walker* (Aberdeen, Md.).

Pedoe, D. On some geometrical inequalities. Math. Gaz. 26, 202–208 (1942). [MF 7912]

Associated with any finite set of n noncollinear points in the x, y-plane with centroid at the origin is a "momental ellipse" $ax^2+2hxy+by^2=1$, such that the sum of the squares of the distances of the points from a diameter of length $2r$ is $1/r^2$. The author expresses in terms of a, b and h the sums of squares of certain areas and lengths associated with the n points, and thereby obtains inequalities connecting these sums. For $n=3$, these give a simple proof of the inequality $(AB^2+BC^2+CA^2)^2\geqq 48\Delta^2$, where Δ is the area of the triangle ABC. Relations between the momental ellipses of two co-centroidal sets of points are used to get similar results. The methods are also extended to three dimensions.
R. J. Walker (Aberdeen, Md.).

Feller, Willy. Some geometric inequalities. Duke Math. J. 9, 885–892 (1942). [MF 7938]

Let Γ be a plane set contained in the unit circle, and suppose the intersection of Γ with any straight line has a measure not exceeding a fixed number $\delta>0$. It is obvious that the measure M of Γ is less than 2δ, and the problem arises to determine the best possible asymptotic estimate for M as $\delta\to 0$. The author constructs a set consisting of a finite number of annuli such that its intersection with any line has length not exceeding δ, but whose area is greater than $2\delta(1-\delta^2\pi^{-2}-\epsilon)$, where $\epsilon>0$ is arbitrarily small, thus proving that the best asymptotic estimate for the area is 2δ.

The author then generalizes this problem to n dimensions and finds the best estimate in the general case; Γ lies in the interior of an $(n-1)$-dimensional unit sphere in n-dimensional space and the length of its intersection with any line of the space is supposed less than or equal to δ. Under these circumstances it is proved that the best asymptotic estimate for the volume of Γ is δV_{n-1}, where V_{n-1} is the volume of the interior of the $(n-1)$-dimensional unit sphere. Actually the author finds this estimate by formulating and solving the following more general analytic problem. Let ω_{n-1} be the volume of the $(n-1)$-dimensional unit sphere, and let $\Phi(r)$ be any nonnegative function for which

$$\int_p^1 \Phi(r)r(r^2-p^2)^{-\frac{1}{2}}dr\leqq\tfrac{1}{2}\delta$$

uniformly in p, $0 \leqq p < 1$. Then

$$M = \omega_{n-1} \int_0^1 \Phi(r) r^{n-1} dr \leqq \delta V_{n-1},$$

and there is equality if $\Phi(r) = \delta\pi^{-1}(1-r^2)^{-\frac{1}{2}}$.

Finally a strikingly simple proof is given of the well-known result that of all convex sets of given volume the sphere has the smallest diameter. *D. C. Spencer.*

Santaló, L. A. Some mean values and inequalities relating to curves on the sphere. Revista Union Mat. Argentina **8**, 113–125 (1942). (Spanish) [MF 7675]

The author projects a regular polyhedron from its center upon its circumscribed sphere and computes (a) the average number of points in which a mobile spherical curve intersects the sides of the projection, (b) the average number of vertices which a mobile domain on the sphere contains. In the second part of this paper he derives some sufficient conditions for a mobile convex domain on the sphere to contain another in its interior. The fact that these conditions cannot be satisfied if the two domains are the same yields at once the isoperimetric inequality on the sphere. Let $R(r)$ be the maximum (minimum) of the radii of all the circles that do not contain (are not contained in) a convex domain of length L and of area F. Since the conditions cannot be satisfied for the domain and the circles of radii R and r, the author obtains the following refinement of the isoperimetric inequality on the sphere of radius one:

$$L^2 + F^2 - 4\pi F \geqq \max\left((L - F\cot\rho/2)^2, ((4\pi - F)\tan\rho/2 - L)^2\right),$$

$\rho = R$ or $\rho = r$. *P. Scherk* (Bloomington, Ind.).

Santaló, Luis A. An integral formula concerning convex figures. Revista Union Mat. Argentina **8**, 165–169 (1942). (Spanish) [MF 8028]

The closed convex curve K shall have a tangent everywhere and shall contain no segments of straight lines. Let L be its length. Let σ be the distance of the two points P_1, P_2 at which a straight line G intersects K, and let α_1 and α_2 ($< \pi$) be the angles between G and K at P_1 and P_2. Then

$$\int \frac{\sigma}{\sin\alpha_1 \sin\alpha_2} dG = \tfrac{1}{2}L^2,$$

where dG denotes the measure of the straight lines G and the integration is extended over all G. The proof follows at once if the arc lengths s_1 and s_2 from a fixed point of K to P_1 and P_2 are chosen as coordinates of G and if

$$dG = \frac{\sin\alpha_1 \sin\alpha_2}{\sigma} ds_1 ds_2$$

is expressed through their differentials. The author also deals with the case where K contains segments of straight lines and with a three-dimensional analogue.

P. Scherk (Bloomington, Ind.).

Alexandroff, A. Existence and uniqueness of a convex surface with a given integral curvature. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 131–134 (1942). [MF 7614]

Let F be a convex surface, O a point in the interior of F and S the unit sphere about O. The spherical image of a set E on F consists, by definition, of the points where the rays with origin O and parallel to the outer normals to supporting planes of F at points of E intersect S. If E is Borel measurable then the spherical image of E has a measure $k(E)$ which is called the integral curvature of E. Call E_S the set on S obtained by projecting E on S. Putting $k(E_S) = k(E)$ we obtain a set function on S. A completely additive set function $k(E_S)$ defined on all Borel measurable sets of S corresponds to the integral curvature of a convex surface if and only if (1) $k(S) = 4\pi$ and (2) for any convex subset E_S of S the relation $k(E_S) < 4\pi - \beta$ holds, where β is the area of the spherical image of the cone projecting E_S from O on E. If, for two convex surfaces F_1 and F_2 which contain O in their interiors, the integral curvatures $k(E)$ are identical, then F_1 and F_2 are homothetic with O as the center of similitude.

The result is extended to complete open convex surfaces F. Consider a plane T on one side of F such that the normal to T intersects F in at most one point. The spherical image of a set E on S is defined as before, but the number $k(E)$ is now associated with the projection E_T of E on T and an additive set function on T is obtained. An additive set function on $k(E_T)$ defined on all Borel sets of T is an integral curvature of a complete open convex surface if and only if $0 < k(T) \leqq 2\pi$. The corresponding uniqueness theorem requires as extra condition that the tangent cones "at infinity" coincide. There are no detailed proofs.

H. Busemann (Chicago, Ill.).

Alexandroff, A. Smoothness of the convex surface of bounded Gaussian curvature. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 195–199 (1942). [MF 8003]

Let F be a convex surface, S the unit sphere about a point O. With a subset E of F we associate the set E_S of those points on S, where the rays with origin O parallel to the outer normals to F at points of E intersect S. Consider any point P on F. If the quotient $m(E_S)/m(E)$ is bounded for every domain E on F which contains P, we say that the Gaussian curvature of F is bounded at P. If the Gaussian curvature is bounded at P, then either F has a tangent plane at P, or P is interior point of a rectilinear edge. Hence, when F is a closed convex surface and the Gaussian curvature is bounded throughout then the surface has a tangent plane everywhere. The first result proves the following conjecture of Cohn-Vossen. A regular surface of positive curvature cannot be bent without self intersection such that it gets an isolated singular point. *H. Busemann.*

Allendoerfer, Carl B. and Weil, André. The Gauss-Bonnet theorem for Riemannian polyhedra. Trans. Amer. Math. Soc. **53**, 101–129 (1943). [MF 7827]

The classical Gauss-Bonnet theorem expresses the curvatura integra (integral of total curvature) of a curved polygon in terms of the angles of the polygon and the geodesic curvatures of its edges. The present paper extends this theorem from the 2 to the n dimensional case. The special case of a closed M^n was recently treated independently by C. B. Allendoerfer [Amer. J. Math. **62**, 243–248 (1940); these Rev. **2**, 20] and W. Fenchel [J. London Math. Soc. **15**, 15–22 (1940); these Rev. **2**, 20], under the assumption that M^n could be imbedded in a Euclidean space E^N.

The main theorem is the following. Let P^n be a Riemannian polyhedron (a bounded manifold) with boundary polyhedra P_λ^r. For any point ζ of any P_λ^r, let $\Gamma(\zeta)$ be the "outer angle" of P^n at ζ, consisting of all unit vectors ξ at ζ (ξ existing in an enlarged P^n with ζ as inner point) such that, for any vector η tangent to P^n at ζ, $\xi_\alpha \eta^\alpha \leqq 0$. (If $P^n \subset E^n$, the set of all $a\xi$ ($0 < a \leqq 1$) clearly fills out a region surrounding P^n, as used in integral geometry.) For each P_λ^r, let $P_{i_1 i_2 j_1 j_2}$ be the components of the curvature tensor of P^n in

directions tangent to P_λ^r, and, for any normal vector ξ to P_λ^r, let $\Lambda_{ij}(\xi)$ be ξ contracted with the coefficients of the second fundamental form of P_λ^r in P^n. Let γ_{ij} be the metric tensor of P_λ^r; set $\gamma=|\gamma_{ij}|$ and

$$\Phi_{rf}(\zeta,\xi)=\frac{\epsilon^{(i)}\epsilon^{(j)}\cdot P_{i_1i_2j_1j_2}\cdots P_{i_{2f-1}i_{2f}j_{2f-1}j_{2f}}\Lambda_{i_{2f+1}j_{2f+1}}(\xi)\cdots\Lambda_{i_rj_r}(\xi)}{2^{2f}f!(r-2f)!\gamma},$$

$$\Psi(\zeta,\xi\,|\,P_\lambda^r)=\pi^{-n/2}\Gamma(n/2)2^{-1}\sum_{f=0}^{[r/2]}\frac{\Phi_{rf}(\zeta,\xi)}{(n-2)(n-4)\cdots(n-2f)}.$$

As a sort of extreme case, define, in P^n,

$$\Psi(z)=(2\pi)^{-n/2}[2^n(n/2)!g]^{-1}\epsilon^{(\nu)}\epsilon^{(\nu)}\cdot R_{\nu_1\nu_2\nu_1\nu_2}R_{\nu_3\nu_4\nu_3\nu_4}\cdots R_{\nu_{n-1}\nu_n\nu_{n-1}\nu_n}$$

for n even, and equal to 0 for n odd. Then, if $\chi'(P^n)$ is the Euler-Poincaré characteristic of the subcomplex of all inner cells of P^n, the theorem states that

$$(-1)^n\chi'(P^n)=\int_{P^n}\Psi(z)dv(z)+\sum_{r=0}^{n-1}\sum_\lambda\int_{P_\lambda^r}dv(\zeta)\int_{\Gamma(\zeta)}\Psi(\zeta,\xi\,|\,P_\lambda^r)d\xi.$$

In case $P^n=M^n$, this reduces to $\chi(M^n)=\int_{M^n}\Psi(z)dv(z)$.

For the proof, P^n is triangulated so that each cell, when approximated to by an analytic one, may be imbedded in some E^N [E. Cartan]; the method of "tubes" as used by Allendoerfer [loc. cit.] is then applied, a calculation of H. Weyl [Amer. J. Math. 61, 461–472 (1939)] playing a fundamental role. Some interesting properties of outer angles are brought out in the proof. *H. Whitney.*

Differential Geometry

Friedlander, F. G. Note on a limit related to the curvatures of two surfaces. Proc. Cambridge Philos. Soc. **38**, 399–400 (1942). [MF 7809]

Let P be a point on a surface S in E^3, where S has elliptic curvature. Let T be a surface through P which has the same tangent plane at P as S, and such that for sufficiently small $h>0$ the parallel surface T_h to T in the direction of the "inner" normal of S at P intersects S in a closed curve which bounds on S an area A_h. The limit of A_h/h for $h\to 0$ is determined as $2\pi\Delta^{-1}$ with

$$\Delta^2=(r_1^{-1}-s_1^{-1})(r_2^{-1}-s_2^{-1})\cos^2\alpha+(r_1^{-1}-s_1^{-1})(r_2^{-1}-s_2^{-1})\sin^2\alpha,$$

where r_1, r_2 and s_1, s_2 are the principal radii of curvature of S and T at P, respectively, and α is the angle between r_1 and s_1. The special case of this result, where T is a plane, is due to Blaschke [Differentialgeometrie I, 3rd ed., J. Springer, Berlin, 1930, p. 120, ex. 10]. *H. Busemann.*

Williams, Martha Hathaway. Ruled surfaces in Euclidean four space. J. Math. Phys. Mass. Inst. Tech. **21**, 144–158 (1942). [MF 7758]

This paper gives some results of an investigation of ruled surfaces immersed in an Euclidean space R_4 of four dimensions. The tensor method is employed. Among the results may be mentioned the following. The Gaussian curvature of the surface is never positive. If a developable surface is defined as being a ruled surface with zero Gaussian curvature, then a ruled surface which is developable is the tangent surface of a twisted curve and conversely. There are no ruled surfaces with constant (nonzero) Gaussian curvature. The only minimal ruled surfaces in R_4 are those in R_3.

The problem of deforming a ruled surface in R_4 into another surface is considered; a partial solution of the problem is obtained. Among the results obtained we may quote the following. A ruled surface can be deformed so that a geodesic becomes a straight line, and the director cone has a fixed projection on R_3 in a single infinity of ways. A ruled surface can be deformed so that a geodesic becomes a straight line and the director cone cuts the hypersphere in a great circle in just one way. *V. G. Grove.*

Santaló, L. A. On certain varieties of the type of a developable in Euclidean space of four dimensions. Publ. Inst. Mat. Univ. Nac. Litoral **4**, 3–42 (1942). (Spanish) [MF 7800]

Let a curve in E_4 be defined by the vector $X=X(s)$ in terms of the arc s and let T and N denote the vectors of unit modulus in the directions $X'(s)$ and $X''(s)$, respectively. Denote by B the unit vector in the E_3 of X', X'', X''' and perpendicular to T and N, and by D a unit vector perpendicular to T, N, B (orientation unspecified). Consider a surface generated by a family of lines having one through each point of such a curve not lying in any E_3. If each generator lies in the corresponding osculating E_3, the tangential surface is the only developable one. If each generator lies in the E_3 determined by T, N, D, there is one developable surface besides the tangential surface. If each generator lies in the E_3 of T, B, D, there is a single parameter developable family besides the tangential surface. A dedevelopable surface (called normal), each of whose generators lies in the corresponding E_3 normal to the curve, is determined by a single generator. In general there is no developable surface besides the tangential one having each of its generators in a face of the corresponding 4-simplex T, N, B, D. Conditions are found under which the generators have an invariant position relative to this simplex, also those under which the developable surfaces are cones, or cylinders. If the curve lies in an E_3, all the generators of a developable surface make the same angle with their corresponding D's.

In the case of three dimensional developable varieties generated by planes, one through each point of the curve (not in any E_3), the generator is the osculating plane if it always contains N. This is the only case in which the generator always contains two of the vectors T, N, B, D. The conditions under which each generator contains T are found similarly for B, D. Each such variety determines a surface to which each generator is tangent. This surface is developable. The conditions under which a three dimensional developable variety is generated by lines through each point on a two dimensional surface in E_4 are found. In general, at each point on the surface there are two directions in which the tangents may serve as generators. The conditions under which the generators are not in the tangent planes are developed, in particular, those under which the generators lie in their corresponding normal planes.

J. L. Dorroh (Baton Rouge, La.).

Vest, M. L. Non-involutorial space transformations associated with a linear congruence. Bull. Amer. Math. Soc. **48**, 874–882 (1942). [MF 7504]

The author considers directrices r and s of a linear congruence and two projective pencils of surfaces $|F_{m+n+1}|$ and

$|F'_{m'+n'+1}|$. The unique line t of the congruence which passes through a generic point $P(y)$ meets the associated F' of $|F'|$ in a residual point $P'(x)$. The latter point is the image of $P(y)$ and serves to define the space transformation to be studied. Denoting the residual base curves of $|F|$ and $|F'|$, other than r and s, by g and g', the author finds that through a point $O_{g'}$ on g' there is a unique line t' of the congruence which lies upon a surface of $|F'|$. The associate surface of $|F|$ meets t' in a point whose locus is a curve $\bar{g}$. Similarly, by use of a point O_g on g, a curve $\bar{g}'$ can be generated. It is shown that r, s, g, g', $\bar{g}$, $\bar{g}'$ are fundamental curves of the space transformation defined above. *N. Coburn.*

Coburn, N. Congruences in unitary space. Trans. Amer. Math. Soc. **53**, 25–40 (1943). [MF 7823]

A study of the properties of a congruence of ∞^{n-1} curves which are imbedded in a unitary space K_n of n dimensions. Necessary and sufficient conditions are determined for the ∞^{n-1} congruence curves to be unitary Euclidean curves or real curves. Two types of orthogonality are defined: (1) ∞^1 hypersurfaces which are completely unitary orthogonal to the congruence curves and (2) ∞^1 hypersurfaces which are semi-unitary orthogonal to the congruence curves. It is shown that in the first case the hypersurfaces admit an intrinsic parameterization and are ∞^1 unitary K_{n-1}. In the second case if the hypersurfaces admit a parameterization they are ∞^1 semi-analytic spaces X_{n-1}. Certain characterizations of congruences are given. *T. Y. Thomas.*

Buchin-Su. Moutard-Čech hyperquadrics associated with a point of a hypersurface. Ann. of Math. (2) **44**, 7–20 (1943). [MF 8071]

The present study is based on the coefficients and coordinates in the development $X^{n+1}=H_\sigma X^\sigma X^\tau+\cdots$ of a general hypersurface, V_n, in projective $(n+1)$-space about a point, O, where $X^{n+1}=0$ is the tangent hyperplane. The fundamental forms, the Darboux pencil and other elements in the Fubini-Čech line of development are explicitly expressed at O in terms of these coefficients. The main investigation centers around a section, V_r, of V_n by an $(r+1)$-plane, $[r+1]$, passing through a fixed r-plane, $[r]$, in the tangent hyperplane and through 0. A new proof is given of a theorem of Čech: as $[r+1]$ turns about $[r]$ the Čech quadric in V_r generates the Moutard-Čech hyperquadric associated with $[r]$. The equation of the latter is derived in the chosen coordinates. It lies in a certain pencil of hyperquadrics which the author defines as the Moutard-Čech pencil associated with $[r]$ in extending a definition in three-space previously given by him and A. Ichida [Jap. J. Math. **10**, 209–216 (1934)]. Five theorems are proved regarding these configurations, one of them a generalization of a theorem of Bompiani in ordinary space. *J. L. Vanderslice.*

Prokofiev, V. Imbedding of two-dimensional spaces of normal projective connectivity into the three-dimensional projective space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 87–88 (1942). [MF 8009]

A two-dimensional projective space with a normal projective connection can be imbedded in a four-dimensional projective space. This note deals with spaces which are projectively equivalent to a surface in a three-dimensional space. It is shown that, if the geodesics are written in the form $d^2y/dx^2=3Edy/dx+F$, then $R^2(k_y-n_x)=0$, where R, k, n are functions of E, F and their derivatives, is the necessary and sufficient condition for imbeddability in three-dimensional projective space. *M. S. Knebelman.*

Galvani, Octave. Sur la réalisation de certains espaces à parallelisme absolu par des congruences de droites. C. R. Acad. Sci. Paris **214**, 337–339 (1942). [MF 7852]

The content is sufficiently described by the title. The notation and terminology is that peculiar to E. Cartan and his students. *T. Y. Thomas* (Los Angeles, Calif.).

Chern, Shiing-shen. On the invariants of contact of curves in a projective space of n dimensions and their geometrical interpretation. Acad. Sinica Science Record **1**, 11–15 (1942). [MF 7475]

Certain contact invariants in projective n-space of two curves having contact of order k are derived by considering the freedom of choice in a reference system which gives them simultaneously canonical form. Then these invariants are exhibited as cross-ratios by generalizing two theorems on osculating plane curves given by C. Segre [Atti. Acad. Naz. Lincei. Rend. (5) **6**, 168–175 (1897)]. This paper and the one which follows are from a new Chinese journal devoted to short scientific articles. *J. L. Vanderslice.*

Chern, Shiing-shen. On a Weyl geometry defined from an $(n-1)$-parameter family of hypersurfaces in a space of n dimensions. Acad. Sinica Science Record **1**, 7–10 (1942). [MF 7476]

This is a generalization to n dimensions of an earlier paper by the author [Ann. of Math. (2) **43**, 545–559 (1942); these Rev. **3**, 309]. The procedure is given only in briefest outline but is a direct extension of that used in the previous three dimensional case. The author has in view the general problem of an m-parameter family of hypersurfaces in n-space, a problem initiated by M. Hachtroudi for the case $m=n$ [Actual. Sci. Ind., no. 565, Hermann et Cie., Paris, 1937]. *J. L. Vanderslice* (College Park, Md.).

Walker, A. G. Note on a distance invariant and the calculation of Ruse's invariant. Proc. Edinburgh Math. Soc. (2) **7**, 16–26 (1942). [MF 7490]

The author studies a distance invariant Φ of two points P and P' in a Riemannian space V_n of n dimensions and shows its application in the study of "harmonic" spaces. If PP' lies on a nonnull geodesic, then Φ is defined as proportional to the $(n-1)$ volume of the cross-section at P' of a thin cone of geodesics passing through P. In the case where PP' lies on a null geodesic, a similar but modified definition is given for Φ. The author sketches a proof of the relation $\Phi=\rho s^{n-1}$, where s is the geodesic distance between P and P' and ρ is the ratio of determinants of the fundamental tensor evaluated at P and P' in normal coordinates. By use of a lemma on matrices, the author is able to expand ρ in powers of s for a fixed point P and a fixed direction PP' at P. Evidently, the coefficients of such an expansion are invariants of the direction vector PP', the curvature tensor and the covariant derivatives of the latter. By definition, spaces for which ρ is independent of the direction of PP' for all points P are called "completely harmonic." A study of the first four invariants in the expansion for ρ furnishes such interesting results as: (1) any space of constant curvature (positive definite metric) is completely harmonic; (2) of the class of V_n conformal to a flat space, only spaces of constant curvature are completely harmonic; (3) of the class of V_4 with signature ± 2, only spaces of constant curvature are completely harmonic. *N. Coburn.*

Gonçalves Miranda, Manuel. Tensor calculus. Anais Fac. Ci. Pôrto **27**, 54 pp. (1942). (Portuguese) [MF 8013]

The pamphlet presents a concise development of the elements of tensor analysis, beginning with definitions and examples of contravariant, covariant and mixed tensors, and continuing with definitions of the fundamental algebraic and infinitesimal operations. Properties of the Christoffel symbols are developed, etc. Some of the usual geometric applications are made, including the curvature of a curve and parallelism. The author concludes with a discussion of the Riemann-Christoffel tensor and the identity of Bianchi.

E. F. Beckenbach (Austin, Tex.).

TOPOLOGY

Goffman, Casper. The approximation of arbitrary biunique transformations. Duke Math. J. **10**, 1–4 (1943). [MF 8095]

A biunique (one-to-one) correspondence κ of a closed square S onto itself is ϵ-approximated by a like-conditioned correspondence κ^* if there exist sets A and B in S, each of relative exterior measure greater than $1-\epsilon$ with respect to S, such that for every point p of A the distance between the images under κ and κ^* is less than ϵ, and the same is true regarding the images under their inverses. The main theorem states that every biunique correspondence of a square into itself can, for every ϵ, be ϵ-approximated by a homeomorphism. The proof is given by ingeniously placing two sets of rectangles, and then using a homeomorphism which carries the rectangles of the one set into those of the other set. A second result, of somewhat similar description, is proved regarding continuity of κ and its inverse, for the case that each carries measurable sets into measurable sets.

A. B. Brown (Flushing, N. Y.).

Schweigert, G. E. Fixed elements and periodic types for homeomorphisms on s. l. c. continua. Proc. Nat. Acad. Sci. U. S. A. **29**, 52–54 (1943). [MF 8024]

With the details of proof to appear elsewhere, the paper extends most of the results obtained previously by Ayres [Fund. Math. **33**, 95–103 (1939); these Rev. **1**, 45] and Whyburn [Analytic Topology, Amer. Math. Soc. Colloquium Publ., v. 28, New York, 1942; these Rev. **4**, 86] on the behavior of cyclic elements of a semi-locally-connected space under transformations having periodic character to certain ortho-homeomorphisms with no periodic character assumed. It is shown that if the homeomorphism $T(S)=S$ possesses any one of the five following conditions it has all five of them: (a1) if C_1 and C_2 are invariant cyclic elements then every cyclic element in the cyclic chain $C(C_1C_2)$ is invariant, (a2) the sum of all invariant cyclic elements of S is a non-empty A-set, (a3) for every pair of distinct cyclic elements E_1 and E_2 such that $T(E_1)=E_2$, the cyclic chain $C(E_1E_2)$ contains one and only one invariant cyclic element, (w1) if X is an invariant A-set in S and $N\neq X$ is an invariant node of X, then there exists a fixed cut point of X, (w2) let H and K be any two continua of S such that $S=H+K$ and $p=H\cdot K$ is a cut point of S. If H and K each meet its image under T then $T(p)=p$. Further, if any homeomorphism $T(S)=S$ has an invariant node $\neq S$ then there is another invariant cyclic element. Finally three necessary and sufficient conditions that a homeomorphism be elementwise periodic except on the end points are given.

W. L. Ayres (Lafayette, Ind.).

Roberts, J. H. and Civin, Paul. Sections of continuous collections. Bull. Amer. Math. Soc. **49**, 142–143 (1943). [MF 7997]

If a separable metric space X is decomposed into compacta, and if this decomposition G is continuous and has an n-dimensional hyperspace, then one can find a closed subset of X whose intersection with any set of G is non-vacuous and consists of at most $n+1$ points.

R. H. Fox (Urbana, Ill.).

Eilenberg, Samuel and Harrold, O. G., Jr. Continua of finite linear measure. I. Amer. J. Math. **65**, 137–146 (1943). [MF 7781]

Let $\alpha\geqq 0$, $\epsilon>0$, and define for a metric space X the function $L_\epsilon^\alpha(X)=$g.l.b. $\sum_{i=1}^{\infty}[\delta(X_i)]^\alpha$ for all decompositions $X=X_1+X_2+\cdots$, $\delta(X_i)<\epsilon$. If no such decomposition exists, set $L_\epsilon^\alpha(X)=+\infty$. The authors then define the α-dimensional measure of X by the equation $L^\alpha(X)=\lim L_\epsilon^\alpha(X)$ as $\epsilon\to 0$. This function $L^\alpha(X)$, which obviously depends upon the particular metrization of X, is evidently defined for every subset A of X. The authors characterize the continua (that is, the nondegenerate compact metric connected spaces) having finite linear measure as the principal result of the paper. Their principal theorem is the following. For every continuum X the following conditions are equivalent: (1) X can be imbedded in the Hilbert cube so as to have a finite linear measure. (2) X is of finite degree. (3) Given any two disjoint closed subsets X_0, X_1 of X, there is a continuous mapping $F(X)=I$, where I is the unit interval, such that (a) $F(x)=i$ for x in X_i, $i=0, 1$, (b) $F^{-1}(y)$ is finite for uncountably many distinct points of I. (4) In (3) condition (b) may be replaced by the condition that $F^{-1}(y)$ is finite for every irrational point of I. In view of these conditions and known results, further equivalent conditions may be added to the ones just stated [see G. T. Whyburn, Amer. J. Math. **57**, 11–18 (1935); O. G. Harrold, Jr., Bull. Amer. Math. Soc. **46**, 951–953 (1940); these Rev. **2**, 178].

D. W. Hall (College Park, Md.).

Clark, C. E. On the join of two complexes. Bull. Amer. Math. Soc. **49**, 126–129 (1943). [MF 7993]

Let σ and τ be p and q-dimensional simplexes, respectively, $p, q=0, 1, \cdots$. The author defines the join of σ and τ, denoted by (σ, τ), to be a $(p+q+1)$-dimensional simplex having a p-dimensional side associated with σ and the opposite side, which is q-dimensional, associated with τ. These sides are not distinguished from σ and τ, respectively. Now consider two complexes K_1 and K_2. Form the complex C consisting of the simplexes σ_α of K_1, the simplexes τ_β of K_2 and the simplexes $(\sigma_\alpha, \tau_\beta)$. The join (K_1, K_2) of the two complexes is defined to be the first barycentric subdivision of C. The author proves that there exists an isomorphism between the $(r+1)$-dimensional Betti group of the join of two complexes and a subgroup of the r-dimensional Betti group of their product. In the case where the complexes are finite an homology basis for this subgroup is pointed out. These results imply a theorem of Freudenthal [Fund. Math. **29**, 145–150 (1937)] concerning the orders of the generators of the $(r+1)$-dimensional Betti group of the join of two finite complexes.

D. W. Hall (College Park, Md.).

Eckmann, Beno. Systeme von Richtungsfeldern in Sphären und stetige Lösungen komplexer linearer Gleichungen. Comment. Math. Helv. **15**, 1–26 (1943).

It is shown that in a sphere S^{4r+1} there is no pair of independent continuous tangent vector functions. The theorem was proved independently by G. W. Whitehead [Ann. of Math. (2) **43**, 132–146 (1942); these Rev. **3**, 197]. The following theorem on complex linear equations is given as a corollary. Let $(u_1, \cdots, u_r)$ range over the set $\sum u_i\bar{u}_i=1$. Then if r is odd, there is no set of continuous functions $v_i=f_i(u_1, \cdots, u_r)$ such that $\sum v_i\bar{v}_i=1$ and $\sum u_i\bar{v}_i=0$.

H. Whitney (Cambridge, Mass.).

Hopf, Heinz. Nachtrag zu der Arbeit Fundamentalgruppe und zweite Bettische Gruppe. Comment. Math. Helv. **15**, 27–32 (1943).

An elementary proof is given for the following theorem which, for the dimension $n=2$, was a corollary of a previous theorem of the author [Comment. Math. Helv. **14**, 257–309 (1942); these Rev. **3**, 316]. Let K_i $(i=1, 2)$ be a connected complex for which the rth homotopy group vanishes $(r=2, \cdots, n-1)$. Let $\mathfrak{B}_i^n$ and $\mathfrak{S}_i^n$ denote their nth Betti groups and groups of "spherical cycles." Then if K_1 and K_2 have isomorphic fundamental groups, $\mathfrak{B}_1^n/\mathfrak{S}_1^n$ and $\mathfrak{B}_2^n/\mathfrak{S}_2^n$ are isomorphic. *H. Whitney* (Cambridge, Mass.).

Hopf, Heinz. Maximale Toroide und singuläre Elemente in geschlossenen Lieschen Gruppen. Comment. Math. Helv. **15**, 59–70 (1943).

The following theorems, appearing in a paper of E. Stiefel [Comment. Math. Helv. **14**, 350–380 (1942); these Rev. **4**, 134] are proved by topological means. Let T be a maximal toroid in the Lie group G. Say $t \in T$ is singular if t is in some maximal toroid $T' \neq T$. The inner automorphisms of G give a group Φ of automorphisms of T. Let $n=\dim G$, $l=\dim T$. Then there are certain subgroups $U_1, \cdots, U_m$ of T, each a toroid or a toroid plus a coset, with $\dim U_i=l-1$, $\dim U_i \cap U_j=l-2$, whose union A is the set of singular elements of T. The group Φ is finite, transforms A into itself, and is generated by m involutions ϕ_i, where ϕ_i is essentially a reflection of T across U_i. *H. Whitney.*

MECHANICS

* Please also read the paragraph on p. 340, Errata, for more information.

Campbell, J. W. Motion re moving axes. Trans. Roy. Soc. Canada. Sect. III. (3) **36**, 1–6 (1942). [MF 7878]

Let $V=V(x, y, z, t)$ represent a vector function referred to a certain fixed rectangular frame F. We next refer both the point (x, y, z) and the vector V to a moving frame F' which coincides with F when $t=t_0$. With respect to F' let the point (x, y, z) be denoted by (x', y', z') and let the vector V have the components V_1', V_2', V_3', which three quantities are regarded as functions of x', y', z' and t. The partial derivatives of these three functions with respect to t evaluated at $t=t_0$ form the components of a vector V_t which the author calls the Newtonian derivative of V. In terms of the partial derivatives of the original function $V(x, y, z, t)$, it is supposedly shown that

$$V_t=\partial V/\partial t+(v\cdot\nabla)V+\omega\times V,$$

where v is the vector velocity of the origin of F' and ω the vector angular velocity of F' when $t=t_0$. The formula is incorrect. There should be an additional term $[(\omega\times r)\cdot\nabla]V$, where r is the vector (x, y, z), and moreover the sign of the last term $\omega\times V$ appears to be wrong. These mistakes are due to a confused formulation of the problem. According to another possible interpretation of the author's formulation, the sign of $\omega\times V$ is correct, the sign of $(v\cdot\nabla)V$ is wrong and there is a missing term $[(r\times\omega)\cdot\nabla]V$. There appears to be no interpretation for which the author's formula is completely correct. *D. C. Lewis* (Durham, N. H.).

Schlapp, Robert. A note on small vibrations. Proc. Edinburgh Math. Soc. (2) **7**, 31–38 (1942). [MF 7492]

The author states: "This note exhibits some old results in what is possibly a new form." Accordingly, it is a purely expository matrix-theoretical treatment of the stable conservative linear dynamical system of n degrees of freedom. The paper includes examples. *D. C. Lewis.*

Bödewadt, U. T. Von den freien Schwingungen eines Kreiselpendels bei endlichen Ausschlägen. II. Z. Angew. Math. Mech. **22**, 34–41 (1942). [MF 7670]

[The first part appeared in the same Z. **20**, 218–234 (1940); these Rev. **2**, 63.] The author's method of obtaining periodic solutions for the equations of motion of the gyroscopic pendulum was explained in the first part, where the exact equations were replaced by simpler equations. In this second part the same method (involving expansion of the solution in Fourier series and the expansion of the frequency and Fourier coefficients in power series with respect to a quantity which measures the amplitudes) is applied to more exact equations. Approximations are still introduced, and indeed the equations actually treated do not, for this reason, admit an energy integral, although the original dynamical system is conservative. The paper concludes with a numerical example. *D. C. Lewis* (Durham, N. H.).

Schrödinger, Erwin. Dynamics and scattering-power of Born's electron. Proc. Roy. Irish Acad. Sect. A. **48**, 91–122 (1942). [MF 7649]

A perturbation consists here in the quadratic terms of the original equations; it creates linear and homogeneous field laws for any modification or modulation superposed on a definite exact solution of the original equations. The proper vibrations of the perturbation field are specified by a 4-vector $\phi_1, \phi_2, \phi_3, \phi_4$ when they are of electric type and by a 4-vector $\psi_1, \psi_2, \psi_3, \psi_4$ when they are of magnetic type. The electric multipole vibration connected with the typical associated Legendre function depends on a function $F_n(r)$ while the corresponding magnetic multipole vibration depends on a function $G_n(r)$. These functions satisfy the differential equation

$$(1+r^4)R''(r)\pm 2r^{-1}R'(r)+[\omega^2(1+r^4)-n(n+1)r^2]R(r)=0,$$

where $e^{i\omega t}$ is the common time factor of the components of the 4-vectors and the upper or lower sign is taken according as the vibration is of the electric or magnetic type. In the discussion of these equations the substitution $z=r^4$ proves useful in giving hypergeometric functions as the leading terms in series in powers of ω^2. Both first and higher approximations are considered. The analysis is developed for the determination of the scattering by an immovable singularity and also by a yielding electron, the latter being represented by imparting to the singularity a harmonic motion in the direction of the electric field of the incident wave. *H. Bateman* (Pasadena, Calif.).

Shor, J. B. On the determination of screw axes in spatial mechanisms. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 267–276 (1941). (Russian. English summary) [MF 7711]

The author studies space linkages, the main problem being the determination of instantaneous axes. In several cases it is shown how these axes can be found using the theory of screws in the form of v. Mises' Motorrechnung. In treating other cases use is made of certain general propositions which are generalizations of propositions well-known in the theory of plane linkages, such as Kennedy's theorem. In these generalizations the conception of a "brush" (i.e. the totality of straight line perpendicular to a given straight line) introduced by Kotelnikof plays an important part.

G. Y. Rainich (Ann Arbor, Mich.).

Relativity, Astronomy

McVittie, G. C. Axiomatic treatment of kinematical relativity. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 210–222 (1942). [MF 7521]

The fundamental results of E. A. Milne's kinematical relativity [J. London Math. Soc. **15**, 44–80 (1940); these Rev. **2**, 25] are derived from nine axioms. All but the last of these deal with the assignment of coordinates and definitions of velocity and acceleration. These are axioms used by Milne. The author uses these to obtain results which Milne and his coworkers had obtained as properties of sets of "equivalent" observers. The ninth axiom defines equivalent observers. It is argued that kinematical relativity cannot be applied to a physical theory without the addition of a further axiom which would enable one to identify a theoretical observer with a terrestrial one.

A. H. Taub (Princeton, N. J.).

de Beauregard, Olivier Costa. Sur la mécanique analytique du point électriquement chargé. C. R. Acad. Sci. Paris **214**, 58–60 (1942). [MF 7835]

The expression for the variation of the (relativistic) impulse-energy in the theory of the point-charge has a term which disappears from the equations under the usual assumptions. Its presence, however, makes possible a more general theory having significance for the Dirac theory of the electron. *D. C. Lewis* (Durham, N. H.).

García, Godofredo. On the problem of n bodies and the motion of a gravitational gas (or a nebulous one). Revista Ci., Lima **44**, 457–469 (1942). (Spanish) [MF 8032]

García, Godofredo. On a new form of the inequalities of K. Sundman in the problem of three bodies. Actas Acad. Ci. Lima **5**, 117–121 (1942). (Spanish) [MF 7979]

García, Godofredo. Generalization of the formula of Lagrange. Generalization of Sundman's inequality, deduced from the homogeneity of the potential and Birkhoff's auxiliary function for the problems of n bodies and of infinitely many bodies (gravitational or nebulous gas). Actas Acad. Ci. Lima **5**, 132–144 (1942). (Spanish) [MF 8031]

García, Godofredo. On a generalization of a new inequality derived from K. Sundman's auxiliary function in the problem of three bodies generalized by George D. Birkhoff. Actas Acad. Ci. Lima **6**, 3–11 (1943). (Spanish) [MF 7980]

The material of the first paper has appeared in an English version [Proc. Nat. Acad. Sci. U. S. A. **28**, 425–427, 428–432 (1942); these Rev. **4**, 57]. In the second paper an inequality of Sundman [Acta Math. **36**, 105–179 (1913), in particular, p. 149; G. D. Birkhoff, Dynamical Systems, Amer. Math. Soc. Colloquium Publ., vol. 9, American Mathematical Society, New York, 1927, ch. 9] is generalized to the form (1) $(R^nH)' \geqq nHR^{n-1}R'$. Here $R^2=\sum m_i r_i^2$, where r_i is the distance of the ith body from the center of gravity and H is Sundman's function [see Birkhoff, loc. cit., p. 267]. In the third paper the program of the first paper (generalization of Sundman's inequality to n bodies and to a continuous medium) is carried through for a generalized n-body problem, in which the potential U is required only to satisfy a certain inequality, and not to be homogeneous of degree -1 [see Birkhoff, loc. cit., p. 292]. The fourth paper is partly a resumé of the third, but contains in addition a generalization of the inequality (1) to the case considered in the third paper. *W. Kaplan* (Ann Arbor, Mich.).

García, Godofredo. On the regularization of the plane problem of three bodies. Actas Acad. Ci. Lima **5**, 101–113 (1942). (Spanish) [MF 8030]

Levi-Civita gave a regularizing transformation of variables for the plane problem of three bodies [Atti Accad. Naz. Lincei. Rend. (5) **24**, 61–76 (1915)]. The present paper gives a new derivation for the expression given by Levi-Civita for the kinetic energy in terms of the new variables.

W. Kaplan (Ann Arbor, Mich.).

Dubiago, A. D. On some questions of motion, structure and disintegration of comets. I. II. Astr. J. Soviet Union [Astr. Zhurnal] **19**, 14–46, **20**, 49–62 (1942). (Russian, English summary) [MF 8084]

The problem of motion of short-periodic comets presents many difficulties. Encke's and Brooks' comets, for instance, both show progressive accelerations of their motion, while Wolf's comet shows a secular retardation. This peculiarity of motion is closely connected with the origin, structure and disintegration of comets. On the assumption that a comet is a swarm of particles (meteorites) it can be studied in the same way as the globular clusters. Hence the results of researches by Eddington and Jeans as well as Schuster's law can be applied to comets. The exact photometric study of the surface brightness of comets may lead then to conclusions on the distribution of their particles.

If m is the mass of a comet, and ρ_1 is such a radius of a comet that the mass outside of it is ϵm, then the mass m' within the radius ρ_1 which can leave it is

$$m'=\frac{13}{21}m\epsilon, \quad \text{where } \epsilon \text{ is small.}$$

The mass outside of this sphere which can enter it is $19m\epsilon/21$. The rate of disintegration of a comet is

$$-\frac{dm}{dt}=-\frac{dm_d}{dt}-\frac{dm_r}{dt},$$

where m_d and m_r refer to particles with direct and retrograde orbits, and

$$-\frac{dm_d}{mdt}=0.0716(4/3\pi)^{1/2}\left(\frac{C}{\rho_d}\right)^{7/2}\delta_e^{1/2}\frac{B_d}{2}$$

$$-\frac{dm_r}{mdt}=0.0716(4/3\pi)^{1/2}\left(\frac{C}{\rho_r}\right)^{7/2}\delta_e^{1/2}\frac{B_r}{2}$$

where $0.08<B_d<0.49$ and $0.29<B_r<0.70$.

As follows from the above equation, the velocity of escaping particles with direct motion is larger than that of particles with retrograde motion. Consequently the outer layer of a comet will in time contain mostly particles with retrograde motion, and so the comet acquires a retrograde rotation. Still later the retrograde moving particles will also be lost and the moment of rotation again becomes zero. Then the whole process will be repeated. For $\rho=4500$ km. and $m=10^{-15}$, the period of semi-disintegration is 20,000 to 70,000 days.

If a comet closely approaches a disturbing body, profound changes occur in its structure. At some given moment previous to the passage of pericentron the Schuster structure is as yet unperturbed. Approaching Jupiter the "pseudo-libration points" are formed and the density is rapidly increasing around these points according to the equation

$$\frac{\delta}{\delta_0}=1+5\frac{r'}{\rho}\frac{\rho^2}{C^2+\rho^2}(\Delta v)^2,$$

where ρ is the parameter of the Jupiter-centered orbit, Δv the arc around Jupiter described in the time $t-t_0$, ρ the distance of the pseudo-libration points from the center of the comet. It is interesting to note that neither the mass of the comet nor Jupiter's mass enter the above equation directly. *V. Krotkov* (Kingston, Ont.).

Hydrodynamics, Acoustics

Kravtchenko, Julien. Sur le problème de représentation conforme de Helmholtz. C. R. Acad. Sci. Paris **214**, 464–466 (1942). [MF 7861]

This note is an extension of the author's previous study [J. Math. Pures Appl. (9) **20**, 35–234, 235–303 (1941); these Rev. **3**, 219; **4**, 58] on the two-dimensional wake problem in incompressible nonviscous flow. In the previous study the contour of the body is such that the slope is continuous except at the nose where the velocity is zero. Here the body is assumed to have a number of vertices. Therefore, the function $\psi(l)$ which gives the inclination of contour of the body, l being the length along the contour, is subjected to the following conditions: (1) $\epsilon\leqq\psi(l)\leqq\pi-\epsilon$, for $\alpha\leqq l\leqq\beta$, with $0<\epsilon<\pi$; (2) $\psi(l)$ is continuous in the interval $\alpha\leqq l\leqq\beta$, except at a finite number (n) of points $l_1, l_2, \cdots, l_n$ where $\psi(l)$ has a discontinuity of the first kind. In each interval $\alpha\leqq l\leqq l_1; \cdots; l_i\leqq l\leqq l_{i+1}; \cdots; l_n\leqq l\leqq\beta$, $\psi(l)$ satisfies a Hölder condition

$$|\psi(l)-\psi(l')|\leqq\text{const.}\ |l-l'|^\nu, \qquad 0<\nu\leqq 1.$$

The author states that, by means of the function theory of Leray and Schauder [Ann. École Norm. (3) **51**, 45–90 (1934)], the wake problem can be shown to have at least one solution. *H. S. Tsien* (Pasadena, Calif.).

Wieghardt, K. Zur Theorie der Wirbelbewegung. Z. Angew. Math. Mech. **22**, 58–60 (1942). [MF 7666]

A. Kneschke [Z. Angew. Math. Mech. **18**, 343 (1938)] has given a minimum principle from which may be obtained the equations of motion of isolated vortices in plane simply connected regions of an ideal fluid. The author rederives Kneschke's result with the help of the Hamilton principle. *E. Reissner* (Cambridge, Mass.).

Schwabe, M. Pressure distribution in nonuniform two-dimensional flow. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1039**, 20 pp. (1943). (7 plates). [MF 7962] Translated from Ing.-Arch. **6** (1935).

Roy, Maurice. Écoulement à symétrie de rotation d'un fluide compressible. C. R. Acad. Sci. Paris **214**, 141–144 (1942). [MF 7840]

The author discusses the absolute irrotational polytropic flow of nonviscous compressible fluid between two discs with axial symmetry. All quantities are assumed to be functions of the distance from the axis of symmetry only. Furthermore, the axial distance or the spacing of the discs is assumed to vary only slowly. Therefore, the flow is essentially two-dimensional. Several interesting relations between the geometry of the flow path and the flow velocity are given. The second part of this note is devoted to some remarks on the application of flow diagrams developed previously by the author [C. R. Assoc. Tech. Maritime Aeronaut. Paris **1939**, 43–151] to the present case. *H. S. Tsien* (Pasadena, Calif.).

Oudart, Adalbert. Théorie des sillages. Problème indéterminé de Levi-Civita et de M. Villat. C. R. Acad. Sci. Paris **214**, 149–151 (1942). [MF 7843]

The author states some theorems concerning the theory of the wake of a body in nonviscous flow. The two-dimensional stream is supposed to be bounded by two walls, not necessarily formed by straight lines. The theorems presented here are extensions of results of J. Kravtchenko [J. Math. Pures Appl. (9) **20**, 35–234, 235–303 (1941); these Rev. **3**, 219; **4**, 58]. If $\phi(s)$ and $\psi(s)$ [see the first of the reviews quoted] are functions characterizing the body and the channel, respectively, a Helmholtz-Kirchhoff system can be constructed with the help of general formulae given by Villat. The author states that: (i) in order that there exists a given mean flow direction Ox at infinity upstream and downstream, ψ must satisfy the condition $\psi(s_1+0)=\psi(s_1-0)=\psi(0)=\psi(\pi)=0$. (ii) A second condition must be satisfied in order that the channel walls have a continuous derivative and asymptotes parallel to Ox at infinity. This condition amounts to a finite width of the channel. Further theorems stated concern the order of tangency of these asymptotes and the points where the free streamlines detach from the body. *H. W. Liepmann* (Pasadena, Calif.).

Thiriot, Karl-Hans. Untersuchung über die Grenzschicht einer Flüssigkeit über einer rotierenden Scheibe bei kleiner Winkelgeschwindigkeitsänderung. Z. Angew. Math. Mech. **22**, 23–28 (1942). [MF 7672]

An air mass of infinite extent and depth is imagined to be at rest relative to a rotating plane earth. At a certain instant the earth's angular velocity changes suddenly from ω_0 to $\omega_1=\omega_0+\omega'$, where ω' is small compared to ω_0. The author writes the Prandtl boundary-layer equations for a viscous incompressible fluid in terms of cylindrical polar coordinates and seeks the steady-state solution. To obtain a first approximation, he assumes the radial and tangential velocity components v_r and v_φ to be of the form $\omega' r f(\zeta)$, and the axial component v_z to be of the form $(\omega'\nu)^{\frac{1}{2}}g(\zeta)$, where $\zeta=z(\omega'/\nu)^{\frac{1}{2}}$ and ν is the kinematic viscosity. Terms involving ω'^2 are then neglected, and a solution of the resulting linear equations is obtained that satisfies the boundary conditions. For a second approximation he employs these results in the second-order (nonlinear) terms in

the Prandtl equations and again is able to construct a solution that satisfies the boundary conditions.

The numerical results are presented in graphical form, showing $v_r/\omega' r$, $v_z/(\nu|\omega'|)^{\frac{1}{2}}$ and $v_\varphi/\omega' r$ as functions of $\eta=z(\omega_0/\nu)^{\frac{1}{2}}$ for the cases $\omega'/\omega_0=1/5, -1/5$. The deviation of the velocity vector (as a function of η) from the classical Ekman spiral is also shown for these values of ω'/ω_0. The streamlines of the flow are shown to be logarithmic spirals, and their exponent for $z=0$ is plotted against ω'/ω_0. The value of $v_z/(\nu|\omega'|)^{\frac{1}{2}}$ for $\eta=\infty$ is also given as a function of ω'/ω_0, and certain theoretical and experimental results for $\omega'/\omega_0=1, -1$ are shown for comparison. *W. R. Sears.*

Dorodnitzyn, A. Laminar boundary layer in compressible fluid. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 213–219 (1942). [MF 7605]

The author considers the problem of the laminar boundary layer for an arbitrary profile in a compressible fluid under the assumptions that there is no heat transfer between the fluid and the body, and that the Prandtl number is equal to unity. He first writes the Prandtl equation of motion for flow in a boundary layer, the equation of continuity and the equation of energy balance. He then introduces the stream function ψ and the new independent variables s and t defined by $\partial s/\partial x=\text{const.}\times p$ and $\partial t/\partial y=\text{const.}\times\rho$, where $p(x)$ and $\rho(x, y)$ are the pressure and density, respectively, and x and y are coordinates measured tangentially and normally to the surface. The resulting differential equation is

$$(14)\quad \frac{\partial F}{\partial t}\frac{\partial^2 F}{\partial s\partial t}-\frac{\partial F}{\partial s}\frac{\partial^2 F}{\partial t^2}=\frac{1-\sigma}{1-\sigma_0}U\frac{dU}{ds}+\frac{\partial}{\partial t}\left[(1-\sigma)^{n-1}\frac{\partial^2 F}{\partial t^2}\right],$$

where $F=\text{const.}\times\psi$, $\sigma=\text{const.}\times u^2$, $\sigma_0=\text{const.}\times V^2=U^2(x)$, u is the velocity tangential to surface, V is the value of u at the outer edge of the boundary layer and n is the empirical exponent in the law of variation of viscosity with temperature. This is integrated first for the case of a flat plate, for which $U=\text{const.}$ The solution is obtained by assuming it to have the form $F=\text{const.}\times s^{\frac{1}{2}}\varphi_0(\tau)$, where $\tau=U^{\frac{1}{2}}t/(2s^{\frac{1}{2}})$. The resulting velocity profiles are shown graphically for a Mach number of 3.05. [These numerical results are in agreement with those of von Kármán and Tsien [J. Aeronaut. Sci. **5**, 227–232 (1938)], calculated under the same assumptions.]

Proceeding to a more general case, the author integrates equation (14) with respect to t from the inner to the outer edge of the boundary layer, that is, from $t=0$ to $t=\delta$. This leads to an equation analogous to von Kármán's well-known integral relation for the incompressible-fluid boundary layer. Following Pohlhausen, he assumes the velocity profile to be expressible in a fourth degree polynomial in t/δ. The resulting differential equation may be solved numerically by a method of successive approximations.

W. R. Sears (Inglewood, Calif.).

Loytzansky, L. G. Laminar boundary layer on a body of revolution. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 166–168 (1942). [MF 8005]

The method outlined in a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 227–232 (1942); these Rev. **4**, 120] is adapted to the flow round a streamlined body of revolution. It is assumed that the presence of a transverse curvature affects only the momentum equation and that the velocity profile of the boundary layer depends on Pohlhausen's parameter in the same manner as for the boundary layer of a cylindrical body. *W. Prager.*

Roy, S. K. On a case of slow viscous flow. Bull. Calcutta Math. Soc. **34**, 109–127 (1942). [MF 7576]

The problem is to find the two-dimensional flow of a viscous incompressible fluid over a dip which has a boundary given by the parametric form

$$y=-(1+\alpha^2)^{-1},\quad x=\alpha[1+(1+\alpha^2)^{-1}].$$

The motion at points far from this dip is approximately that of uniform shear. The interesting point of the problem is to find whether there is a critical Reynolds number beyond which a vortex with closed stream lines is formed near the dip. Similar problems with various boundaries were investigated by W. R. Dean [Proc. Cambridge Philos. Soc. **32**, 598–613 (1936); **36**, 300–313 (1940); these Rev. **2**, 168]. The boundaries used by Dean all involve a sharp edge and he found that, although the size and shape of the vortex do depend upon the Reynolds number, the vortex appears for all small Reynolds numbers and there is no critical Reynolds number as defined above.

The present author solved the problem by a method developed by Dean which essentially is to expand the solution in a series of increasing powers of Reynolds number. The first approximation (zero order in Reynolds number) is the familiar biharmonic function. This is shown to be free of vortex. The second approximation (first order in Reynolds number), which partially accounts for the inertia effect, does give vortex near the dip. However, the question of critical Reynolds number is not discussed by the author.

H. S. Tsien (Pasadena, Calif.).

Seth, B. R. Viscous solutions obtained by superposition of effects. Proc. Indian Acad. Sci., Sect. A. **16**, 193–195 (1942). [MF 7794]

Suppose that a solid moves slowly through a viscous fluid, so that the inertia terms may be neglected in the equations of motion. The author remarks that the solution can be represented as the sum of two solutions: (a) irrotational solution obtained by assuming that the body moves through a nonviscous fluid with a particular velocity and (b) solution due to a concentrated force applied in the direction of motion of a solid in an infinite viscous fluid. The sphere and the circular cylinder are discussed specifically.

A. E. Heins (Cambridge, Mass.).

Rocard, Yves et Véron, Marcel. Sur la consection vise d'un fluide s'écoulant en régime laminaire le long d'une plaque. C. R. Acad. Sci. Paris **214**, 301–304 (1942). [MF 7851]

The authors treat the heat conduction by forced convection along a flat plate. If the flow is assumed to be parallel to the plate represented by the x-axis, the equation for the temperature T is given by

$$c\rho u\partial T/\partial x-\lambda\partial^2 T/\partial y^2=q(u, x),$$

where c is the specific heat at constant pressure per unit mass of fluid, ρ the density, u the velocity, λ the heat conductivity and q the heat generated per unit time per unit volume. To solve this equation, the authors cut the flow region into small strips in the y-direction, Δx in width. In each strip it is assumed that $xq=\bar{m}uQ_f$, where $\bar{m}$ is a number and Q_f the heat potential of the fluid per unit volume. Both are assumed to be constant in this strip. By introducing the parameter $z=y/x^{\frac{1}{2}}$, and by using the well-known approximations to the Blasius solution for the boundary layer along a flat plate, the equation for T can be integrated easily. The result is expressed by the local

heat conduction coefficient α_x defined by

$$\alpha_x=\frac{\lambda}{T_\infty-\theta_0}\left(\frac{\partial T}{\partial y}\right)_{y=0},$$

where T_∞ is the absolute temperature far from the plate and θ_0 is the plate temperature. For $c\rho\nu/\lambda\cong 0.76$, $\nu=N_u$, the calculation gives

$$\alpha_x=0.333\lambda(u_\infty/\nu x)^{\frac{1}{2}}[1+2.3\bar{m}Q_f/c\rho(\overline{T_\infty-\theta_0})],$$

where u_∞ is u far from the plate. This result can be compared with that of Pohlhausen [Z. Angew. Math. Mech. 1, 115 (1921)], who assumed $Q_f=0$. *H. S. Tsien.*

Mohr, Ernst. Bemerkung zur Kirchhoffschen Plattenströmung. Z. Angew. Math. Mech. 22, 56 (1942). [MF 7668]

Mohr, Ernst. Die Bernoullische Gleichung und die Allseitigkeit des Druckes. Z. Angew. Math. Mech. 21, 380–381 (1941). [MF 7655]

Heinrich, G. Über Strömungen von Schäumen. Z. Angew. Math. Mech. 22, 117–118 (1942). [MF 7660]

The problem of flow of foams is studied by the author under the following assumptions: (1) The mixing of the liquid, which is considered as incompressible, and the gas is very thorough and stable, so that the mixture can be considered as one fluid with properties representing the average of liquid and gas. (2) The gas is a perfect gas. The author then calculates the density ρ, the equation of state and the adiabatic relation between pressure p and density ρ for the fluid, all involving the weight ratio μ of gas and liquid as a parameter. The computation of the velocity of sound a, defined as $a^2=dp/d\rho$, shows the interesting fact that, for the mixture of a heavy liquid and a light gas, the sound velocity is a minimum when the volume of liquid and the volume of gas in the mixture are equal. This minimum is only a fraction of the velocity of sound of the gas alone. *H. S. Tsien* (Pasadena, Calif.).

Krienes, K. und Schade, Th. Theorie der schwingenden kreisförmigen Tragfläche auf potentialtheoretischer Grundlage. II. Numerischer Teil. Luftfahrtforschung 19, 282–291 (1942). [MF 7902]

Supplementing a previous theoretical paper by Th. Schade [Luftfahrtforschung 17, 387–400 (1940); these Rev. 3, 285], the present paper contains formulae, tables and diagrams facilitating the application of the theory. *W. Prager.*

Rosenblatt, Alfred. On the theorem of Kutta-Joukowsky in aerodynamics. Revista Ci., Lima 44, 259–292 (1942). (Spanish) (1 plate) [MF 7544]

The scope of the present paper is the same as that of an earlier one [Actas Acad. Ci. Lima 5, 33–43 (1942); these Rev. 4, 61] but the mathematical details are presented in greater detail. In both papers the well-known formulas for the force and moment on a profile in two-dimensional potential flow are expressed in terms of the residues of w^2 and w^2z, respectively, where w is the derivative of the complex potential in the z plane. [It is the reviewer's opinion that the author's formulas in terms of the residues are incorrect. Specifically, he subtracts from the results of integration around a large contour the contributions from singularities at the profile, rather than deducting only contributions from possible singularities in the fluid outside the profile. This leads, for example, to the result that the force on a plane airfoil is directed normal to the airfoil rather than normal to the undisturbed velocity.] *W. R. Sears.*

Hantzsche, W. und Wendt, H. Der Kompressibilitätseinfluss für dünne wenig gekrümmte Profile bei Unterschallgeschwindigkeit. Z. Angew. Math. Mech. 22, 72–86 (1942). [MF 7664]

This paper constitutes an extension of the well-known Prandtl-Glauert approximation for the subsonic, two-dimensional flow of a compressible fluid about a thin profile. The stream function is assumed to be developed in the form

$$(1)\qquad \psi/U=-y+g(x,y)\lambda+h(x,y)\lambda^2+f(x,y)\lambda^3+\cdots,$$

where λ is a parameter characterizing the deviation of the profile from the x axis. This expression is substituted into the nonlinear differential equation of motion for steady, irrotational motion, and coefficients of successive powers of λ are equated. This leads to a series of differential equations for $g, h, f, \cdots$. With the notation $\psi^{(1)}/U=-y+g\lambda$, $\psi^{(2)}/U=-y+g\lambda+h\lambda^2$, etc., these are

$$(2)\qquad \psi^{(1)}_{xx}\cdot(1-\alpha^2)+\psi^{(1)}_{yy}=0,$$

$$(3)\qquad \psi^{(2)}_{xx}\cdot(1-\alpha^2)+\psi^{(2)}_{yy}$$
$$=2\alpha^2[(\psi^{(1)}_x/U)\psi^{(1)}_{xy}-\mu^2(1+\psi^{(1)}_y/U)\psi^{(1)}_{xx}]$$
$$-\alpha^4\mu^2(\kappa-1)\psi^{(1)}_{xx}(1+\psi^{(1)}_y/U),$$

where α is the Mach number of the undisturbed flow, μ denotes $(1-\alpha^2)^{-\frac{1}{2}}$ and κ is the adiabatic exponent. These equations are solved by successive approximations. A solution $\psi^{(1)}$ is first obtained for (2) (after making the substitutions $\xi=x$ and $\eta=y/\mu$) that satisfies the boundary conditions that actually apply to the total solution ψ. This is the Prandtl-Glauert approximation; it is then inserted in the right side of (3) and a solution is carried out for the second approximation $\psi^{(2)}$, again satisfying the boundary conditions of ψ. This solution is accomplished after transforming the region outside the profile in the ξ, η plane into the region exterior to a circle. Higher approximations are not carried out in this paper, nor do the authors supply a proof of convergency of the process of successive approximations.

In this paper the first and second approximations are calculated for the following profiles: circular arc, ellipse and lenticular profile, all at no angle of attack, and the symmetrical Joukowsky profiles at arbitrary angles. In the first three cases it is found that the maximum local velocity ratio calculated according to the second approximation increases more rapidly with increasing Mach number α than according to the Prandtl-Glauert method. The lift coefficient of the Joukowsky profile also increases more rapidly with α in the second approximation than indicated by the first approximation; this increment arising from the second approximation is proportional to the thickness ratio but independent of the angle of attack, so that the first and second approximations are identical for an inclined flat plate. *W. R. Sears* (Inglewood, Calif.).

Malavard, Lucien. Relations de réciprocité dans la théorie de l'aile portante. C. R. Acad. Sci. Paris 214, 105–107 (1942). [MF 7837]

Let the local circulation, chords, angles of attack and profile coefficient of two wings of like span be $c(x)$, $l(x)$, $i(x)$, $k(x)$ and $C(x)$, $L(x)$, $I(x)$, $K(x)$, respectively, and let $p(x)k(x)l(x)=P(x)K(x)L(x)=1$. If ϕ, Φ are the velocity

potentials of the two systems of free eddies, use of an integral of $\phi d\Phi/dn - \Phi d\phi/dn$ along the x-axis and of wing theory gives the formula

$$(1) \qquad \int_{-b}^{b} (cI - C\bar{\imath})dx = \int_{-b}^{b} cC(P-p)dx.$$

When $P=p$ the foregoing relation indicates that $c(x) = \int_{-b}^{b} g(x,t)\bar{\imath}(t)dt$, where $g(x,t)=g(t,x)$. This relation can also be derived from Gebelein's theoretical solution and gives a reason for the symmetry of the coefficients in the infinite set of linear equations used by Lotz. When $P \neq p$ an integral equation

$$G(x,y) = g(y,x) + \int_{-b}^{b} [p(t) - P(t)]G(t,y)g(t,x)dt$$

is found for the determination of G when g is known. A number of facts relating to lift coefficients, incidence for zero lift, etc., are inferred from the relation (1). Indications for numerical work are given and mention is made of some tables $c(x)$ to be calculated rapidly for any given form of $\bar{\imath}(x)$. In order to pass from a wing with function $p(x)$ to one with function $P(x)$ a method of approximation is used in which the integrals are replaced by sums and the resulting set of linear equations

$$C(x_m) = c(x_m) + \sum c_n(x_n)C(x_n)[p(x_n) - P(x_n)],$$

$$c(x_m) = \sum c_n(x_m)\bar{\imath}(x_n)$$

is solved by a method of successive approximations. With intervals of $b/10$ and $b/20$ for wings of elliptic and trapezoidal forms $C(x_m)$ can be found with an error of less than 1 percent, the local error being about a tenth of this except at the tips. *H. Bateman* (Pasadena, Calif.).

Schmidt, Wilhelm. Turbulente Ausbreitung eines Stromes erhitzter Luft. II. Z. Angew. Math. Mech. **21**, 351–363 (1941). [MF 7657]

[The first part appeared in the same Z. **21**, 265–278 (1941).] The author investigates the turbulent flow of an axially symmetric jet of heated air. The jet is assumed to have no initial momentum; the motion is due to the difference in density of the heated air in the jet and the surrounding unheated air. The problem is treated on the basis of Prandtl's momentum transfer theory. The mixing length is assumed to be proportional to the width of the jet. [Reviewer's remark: It is possible that a justification of this assumption for the case of variable density has been given in the first part of this paper, which was not available to the reviewer.]

To find the rate of change of the temperature D and the velocity u with the distance from the origin, the author writes $\vartheta = Dx^{-m}$, $u = Wx^{p}$, with constant D and W. Then p and m are determined from (1) the assumption of constant heat transport through every cross section, (2) the conservation of momentum. The result is $m=5/3$, $p=-1/3$. Hence, with $\eta = y/x$, one can write $u = x^{-1/3}f(\eta)$, $\vartheta = x^{-1/3}\Theta(\eta)$. Next u, v and D are determined from the equations of motion, the continuity equation and the heat transport equation. The solutions are given in series form. The result shows that the temperature decreases more rapidly with increasing distance from the axis than the velocity. The results of measurements of the velocity and temperature distribution are given. The calculated values of p and m agree with the experiment. The agreement between the measured and computed temperature and velocity profiles is not very satisfactory. *H. W. Liepmann.*

Haurwitz, B. The applications of mathematics in meteorology. Amer. Math. Monthly **50**, 77–84 (1943). [MF 8050]

Polubarinova-Kochina, P. J. On filtration under hydrotechnical structures in a stratified medium. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 287–302 (1941). (Russian. English summary) [MF 7714]

The author considers the plane problem of the motion of subterranean water in a medium composed of horizontal strata of ground of various permeabilities, the foundation of the structure forming an arc of an analytic curve. The problem is solved by replacing the contour of the foundation of the structure by a system of point vortices and choosing a density of these vortices that will satisfy the given boundary conditions; the velocity set up by vortices in a stratified medium is determined by means of an extension of a method used by B. K. Riesenkampf for the case of a point vortex in a two-layered medium. The integral equation of the problem is now set up, and finally an expansion solution determined according to the procedure of L. I. Sedov [Uspekhi Matem. Nauk **6**, 120–182 (1939); these Rev. **1**, 89]. It is stated that for practical purposes the first two terms of the series probably will be sufficient. *E. F. Beckenbach* (Austin, Tex.).

Theory of Elasticity

Dávila C., Rafael. Partition of energy in elastic waves obtained by reflection and refraction. Revista Ci., Lima **44**, 365–376 (1942). (Spanish) [MF 7543]

This article is a summary, in Spanish translation, with a few printing mistakes and misinterpretations, published without credit or reference, of the seventh chapter of my "Geodynamics" [John Wiley, New York, 1936]. *J. B. Macelwane, S.J.* (Saint Louis, Mo.).

Saibel, Edward. A general method of approximation to the influence function of an elastic system. J. Franklin Inst. **234**, 535–548 (1942). [MF 7539]

The method employed consists in replacing the differential equation for the free vibration of the system by a set of difference equations written for each point of a uniform network laid over the system. This leads to a system of linear homogeneous equations. If the reciprocal of the matrix of the coefficients of this linear system is formed, the element which occupies the jth row and kth column in the reciprocal matrix is the approximate displacement of the system at the point j in the network due to unit load at the point k. The reciprocal matrix thus exhibits the values of the influence function, a Green's function, at n points. The justification of the procedure is based on the fact that the influence function is the kernel of the integral equation for a characteristic function representing a natural mode of vibration. *H. W. March* (Madison, Wis.).

Epstein, Paul S. On the theory of elastic vibrations in plates and shells. J. Math. Phys. Mass. Inst. Tech. **21**, 198–209 (1942). [MF 7762]

A general method is outlined for systematically developing the basic equations governing the natural vibrations of plates and shells with respect to terms of increasing order in the thickness of the plate or shell. The treatment is independent of the classical assumptions, according to which

the normals to the undeformed middle surface are deformed into the normals to the deformed middle surface and the stress normal to the middle surface is negligible. The procedure is, however, based on the assumption that the stresses acting on elements parallel to the middle surface can be represented by convergent series of integral powers of the parameter representing distance from the middle surface [G. D. Birkhoff, Philos. Mag. (6) **43**, 953–962 (1922)], and is essentially similar to the methods used by Sibert [Trans. Amer. Math. Soc. **33**, 329–369 (1931)] and others in dealing with the state of stress in statically loaded flat plates. (As was pointed out by Goodier [Trans. Roy. Soc. Canada. Sect. III (3) **32**, 65–88 (1938)] in connection with expansions of a similar type, the solutions obtained under this assumption are in general incapable of representing the true state of stress in the neighborhood of the edges of the plate or shell unless the forces acting on the edges are distributed over the thickness according to laws prescribed by the theory.) The known system of equations involving terms of zero order in the thickness (the basis of the so-called membrane theory) is presented for the general case, while the lengthy calculations necessary for the determination of terms of second order are carried out only for the case of a right circular cylindrical shell and for the limiting case of a flat plate. The theory is applied to the determination of the critical frequencies for flexural and torsional vibrations of the simplest type. *F. B. Hildebrand.*

Duncan, W. J. Free and forced oscillations of continuous beams: treatment by the admittance method. Philos. Mag. (7) **34**, 49–63 (1943). [MF 8019]

This paper deals with the problem of determining the steady oscillation of a continuous beam subjected to a distribution of impressed periodic forces in the transverse direction. It is shown that by using the concept of admittance [Carter, Reports and Memoranda of Aeronautical Research Committee, no. 1758, July, 1936] a theorem of three moments, entirely analogous to the corresponding theorem in the static problem, can be formulated in a direct and general way. The formulation leads to a set of linear equations, each equation involving the bending moments at three successive supports weighted by certain influence coefficients (admittances) which depend on the frequency of the impressed force, the section characteristics of the beam, and the nature of the end loading. The constant term present in each equation depends, in addition, on the nature and distribution of the transverse loading. In the case of free vibration the constant terms are absent and the eliminant of the complete set of equations is the frequency equation of the beam.

Expressions for the admittances and for the constant terms are listed for beams of piecewise constant flexural rigidity, with and without end thrust or tension.

F. B. Hildebrand (Cambridge, Mass.).

Dschanelidse, G. J. und Radzig, M. A. Die dynamische Stabilität eines Ringes unter der Wirkung der normalen periodischen Belastung. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) **4**, 55–60 (1940). (Russian. German summary) [MF 7882]

The paper deals with the dynamic stability of a thin circular ring loaded with a uniformly distributed force of the type $f=f_0\cos\omega t$. Under the assumptions of the usual approximations the component u of the displacement and the component γ of the rotation satisfy linear sixth order partial differential equations in t and θ (θ = polar coordinate). Putting $u, \gamma = T(t)F(\theta)$, F may be represented as a Fourier series and T as a solution of an equation of Mathieu. [A misprint: the term $-B^{-1}\rho a^4\partial^4 u/\partial t^2\partial u^2$ is missing on the right hand side of eq. (7).] *I. Opatowski* (Chicago, Ill.).

Sadowsky, M. A. Equiareal patterns. Amer. Math. Monthly **50**, 35–40 (1943). [MF 7944]

In a previous paper [J. Appl. Mech. **8**, A-74–A-76 (1941); these Rev. **3**, 96] the author introduced the term "equiareal pattern" for the pattern formed by the parametric curves of an orthogonal curvilinear coordinate system for which the coefficients of the first fundamental form fulfill the condition $EG=1$. [J. Boussinesq [C. R. Acad. Sci. Paris **74**, 242 (1872)] has shown that the stress trajectories of the plane problem of plasticity form such a pattern and has given an analytical representation of equiareal patterns [ibid., p. 318].] In the present paper a special case is studied which can most easily be obtained in the following way. According to the mechanical interpretation mentioned above, the lines intersecting the parametric curves of an equiareal pattern at $\pm 45°$ form the lines of maximum shearing stress of a plane plastic stress distribution. The patterns formed by these lines have been studied by H. Hencky, L. Prandtl, E. Schmidt, C. Carathéodory and others. It is known that a family of parallel plane curves together with their common normals form such a pattern. The lines intersecting the parallel curves under $\pm 45°$ form the type of equiareal pattern studied in the present paper. This pattern is shown to possess an interesting equimileage property. *W. Prager* (Providence, R. I.).

Ghosh, N. N. A matrix method of analysing strain and stress in hyperspace. Bull. Calcutta Math. Soc. **34**, 143–152 (1942). [MF 7580]

In a previous paper [same Bull. **32**, 109–120 (1940); these Rev. **3**, 191] the author expressed an n-vector by an $(n+1)\times(n+1)$ matrix. This notation is used in the present paper to obtain matrix expressions for relative displacement and other elements of the theory of strain and stress in n dimensions. The discussion includes the reduction of a symmetric matrix to the sum of matrices of the form UA^2U, where A is a vector and U a matrix with a leading diagonal $0, 1, 1, \cdots$ and zeros elsewhere. *J. L. Synge.*

Goguel, Jean. Sur la modification de contraintes intérieures par relaxation. C. R. Acad. Sci. Paris **214**, 410–412 (1942). [MF 7855]

Goguel, Jean. Calcul des contraintes, dans l'hypothèse de la relaxation complète. C. R. Acad. Sci. Paris **214**, 470–471 (1942). [MF 7863]

On the basis of a minimum principle is it concluded that, with regard to the determination of the state of stress in the interior of the earth, the usual hypotheses [see, for example, Love, Some Problems of Geodynamics, Cambridge, 1911, p. 12] are equivalent to the assumption that the true state of stress consists in the superposition of a state of stress, in which the deformations satisfy the usual stress-strain relationships, and a state of initial stress wherein the permanent deformations are infinitesimal and governed by the laws of viscous deformation. *F. B. Hildebrand.*

Reinitzhuber, F. Beitrag zur Berechnung gedrückter, dünnwandiger Profile oberhalb der Beulgrenze. Luftfahrtforschung **19**, 240–247 (1942). [MF 7441]

An analysis is made of the stresses and deformations in a thin-walled tube of quadratic cross-section which carries

an axial load. The walls of the tube are considered as thin flat plates with finite deflection. The nonlinear differential equations corresponding to this assumption are solved approximately by the use of the direct methods of the calculus of variations. Further approximations are made in the formulation of the boundary conditions for the plates. Diagrams are given for the calculated stresses under a load which is twice the critical (crinkling) load of the tube. It is found that the value of the deflection-load ratio after crinkling is 4/10 of the value of this ratio before crinkling. In the range between critical load and twice the critical load this result agrees with experimental results of the author. Crushing of the tubes occurred in one case at 2.3 times the critical load and in another case at 3.1 times the critical load. A theoretical determination of the crushing strength has not been attempted. *E. Reissner.*

Platrier, Charles. Au sujet des cisaillements superficiels d'un prisme. C. R. Acad. Sci. Paris **214**, 413–414 (1942). [MF 7856]

Some simple solutions of problems concerning the state of stress in hollow right circular prisms acted upon by surface shearing forces are listed. *F. B. Hildebrand.*

Lekhnitsky, S. G. Equilibrium of an anisotropic cantilever beam. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 3–18 (1942). (Russian. English summary) [MF 7731]

The paper deals with the stresses and displacements of an anisotropic cantilever beam loaded at the free end by a transverse force. The line of action of this force is supposed to coincide with a principal axis of inertia of the cross section of the prismatic beam. The anisotropy of the material is assumed to be of the most general kind, characterized by 21 constants of elasticity. *W. Prager.*

Dudley, Leslie P. Built-in and continuous beams. Aircraft Engrg. **14**, 306–309, 319 (1942). [MF 7559]

A systematic derivation of known results and a discussion of their application to problems in aircraft engineering. *H. W. March* (Madison, Wis.).

Strandhagen, A. G. Application of Maclaurin series to beams under simultaneous transverse and axial loading. J. Aeronaut. Sci. **9**, 529–532 (1942). [MF 7557]

Ruchadze, A. K. Biegung nahezu prismatischer Stäbe. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 123–138 (1942). (Russian. German summary) [MF 7744]

The author studies the bending of "almost prismatic" bars by transverse forces. (A body bounded by a surface of the equation $F[x(1-kz), y(1-kz)]=0$ is called almost prismatic if k is small as compared with 1.) The solution developed in the present paper is approximate in so far as all terms involving k^2 or higher powers of k are consistently neglected. *W. Prager* (Providence, R. I.).

Wolf, K. Kreiszylindrische Behälter auf nachgiebiger Unterlage. Ing.-Arch. **12**, 259–264 (1941). [MF 6427]

Explicit expressions are derived for deflection and stresses in a circular water tank, the bottom plate of which is resting on an elastic foundation, by determining the constants of integration in the known solutions of the equations for the rotationally symmetrical deflections of a circular cylindrical shell and of a circular flat plate on an elastic foundation. *E. Reissner* (Cambridge, Mass.).

Sokolowsky, W. W. Ueber ein Problem der elastisch-plastischen Torsion. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 241–246 (1942). (Russian. German summary) [MF 7754]

Kachanov, L. M. Variation principles for plastic-elastic solids. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 187–196 (1942). (Russian. English summary) [MF 7748]

The author establishes the variational principles of statistics and dynamics of certain solids of nonlinear elasticity studied in the note reviewed ~~above~~ on p. 232. *W. Prager.*

Shevchenko, K. N. Application of the theory of plasticity to the rolling of metals. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 439–452 (1941). (Russian. English summary) [MF 7726]

Schevtchenko, K. N. On the distribution of stresses in a rolled bar. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 381–394 (1942). (Russian. English summary) [MF 7589]

The papers are concerned with the stress distribution in a rolled plate in the neighborhood of the rolls. The problem is treated as two-dimensional and the rolled material is supposed to yield under a constant maximum shearing stress. The distribution of the (frictional) shearing stress between the rolls and the plate is determined from the condition that all along the surfaces of contact with the rolls the plate is in the plastic state. The first paper deals with the case of a thick plate where the plastic regions developing at the rolls do not extend across the entire thickness of the plate. The second paper is devoted to the case of a thin plate where all the material passing between the rolls is in the plastic state. *W. Prager.*

Ishlinsky, A. J. Plane deformation when hardening takes place according to the linear law. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 57–70 (1941). (Russian. English summary) [MF 7701]

Korenev, B. G. On the application of compensating loads. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 91–94 (1942). (Russian. English summary) [MF 7739]

Drymael, Jean. The design of trusses and its influence on weight and stiffness. J. Royal Aeronaut. Soc. **46**, 297–308 (1942). [MF 7562]

BIOGRAPHICAL NOTE

Quarterly of Applied Mathematics.

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